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MESSENGER OF MATHEMATICS.

THE INTEGRATION OF INFINITE SERIES.

By *T. J. I'a. Bromwich*, Queen's College, Galway.

Mr. G. H. Hardy, in the December and January numbers of the *Messenger* (Vol. XXXV., 1905, p. 126), has given a test for extending the rule for term-by-term integration to some cases when the series ceases to converge at the upper limit of the integral.*

Although this test is very simple in many practical cases, yet it does not cover some cases of a comparatively elementary type (see §§ 2, 4 below). In § 3 will be found two forms of test which have proved useful in dealing with special power-series not satisfying Hardy's condition (iii); and § 5 contains another test, more general in appearance than those of § 3, but this test has generally not been so convenient as the other two. § 1 contains an alternative investigation of Hardy's test.

1. If $I_{\mu,\nu}$ is a function of μ, ν which has the monotonic property of never decreasing when μ and ν increase to infinity, it is known† that the equations

$$(1) \qquad \lim_{(\mu)(\nu)} I_{\mu,\nu} = \lim_{(\nu)(\mu)} I_{\mu,\nu} = \lim_{(\mu,\nu)} I_{\mu,\nu}$$

are valid; but of course this does not imply that the common value (1) is necessarily *finite*.

Now, if $F(x, \nu)$ is never negative in the interval (a, A) and never decreases as ν increases, the integral

$$I_{\mu,\nu} = \int_a^{A-1/\mu} F(x, \nu)\, dx$$

* Or when the upper limit is a point of non-uniform convergence.

† Pringsheim, *Münchener Sitzungsberichte*, Bd. XXVII., 1897, p. 101. The notation adopted is explained by

$$\lim_{(\mu)(\nu)} = \lim_{\mu=\infty} \left(\lim_{\nu=\infty}\right), \quad \lim_{(\mu,\nu)} = \lim_{\mu,\nu=\infty}$$

(see Bromwich and Hardy, *Proc. Lond. Math. Soc.*, series 2, Vol. II., 1904, p. 161).

satisfies the monotonic condition, so that in particular we have

$$(2) \quad \lim_{(\mu)(\nu)} \int_a^{A-1/\mu} F(x, \nu)\, dx = \lim_{(\nu)(\mu)} \int_a^{A-1/\mu} F(x, \nu)\, dx.$$

Now, under Hardy's conditions,* let us write

$$F(x, \nu) = f(x) \sum_0^\nu |\phi_n(x)|, \quad G(x, \nu) = f(x) \sum_0^\nu [|\phi_n(x)| + \phi_n(x)],$$

where $f(x)$, being continuous, may be supposed positive from $x = a$ to $x = A$ without loss of generality.

With this interpretation it is clear that $F(x, \nu)$ satisfies the conditions for the validity of (2): and in like manner we find

$$(2') \quad \lim_{(\mu)(\nu)} \int_a^{A-1/\mu} G(x, \nu)\, dx = \lim_{(\nu)(\mu)} \int_a^{A-1/\mu} G(x, \nu)\, dx.$$

We shall now take the difference between the two equations (2′) and (2): but before we can be assured of the legitimacy of this step, it must be proved that the limits (2) and (2′) are both *finite*. Now, if L is the finite upper limit to $|f(x)|$ in the interval (a, A), we have

$$F(x, \nu) < L\bar{\phi}(x) \quad \text{and} \quad G(x, \nu) < 2L\bar{\phi}(x),$$

so that
$$\int_a^{A-1/\mu} F(x, \nu)\, dx < \int_a^A L\bar{\phi}(x)\, dx,$$

and
$$\int_a^{A-1/\mu} G(x, \nu)\, dx < 2\int_a^A L\bar{\phi}(x)\, dx.$$

The integral $\int_a^A \bar{\phi}(x)\, dx$ being supposed convergent, the last pair of inequalities show that the limits (2) and (2′) are both finite.

We may therefore subtract (2) from (2′) without fear of error, and we obtain

$$(3) \quad \lim_{(\mu)(\nu)} \int_a^{A-1/\mu} f(x) \left[\sum_0^\nu \phi_n(x)\right] dx = \lim_{(\nu)(\mu)} \int_a^{A-1/\mu} f(x) \left[\sum_0^\nu \phi_n(x)\right] dx.$$

Now, since $\sum_0^\infty \phi_n(x)$ converges uniformly to the sum $\phi(x)$

* These are :—(i) $f(x)$ is continuous throughout (a, A); (ii) $\phi(x)$ can be expanded in a series of continuous functions $\Sigma\phi_n(x)$, uniformly convergent throughout $(a, A-\epsilon)$; (iii) $\int_a^A \bar{\phi}(x)\, dx$ is convergent, where $\bar{\phi}(x) = \Sigma\, |\phi_n(x)|$.

throughout the interval $(a, A - 1/\mu)$, we have

$$\lim_{(\nu)} \int_a^{A-1/\mu} f(x) \left[\sum_0^\nu \phi_n(x)\right] dx = \int_a^{A-1/\mu} f(x) \phi(x) \, dx,$$

so that the limit on the left-hand side in (3) is equal to

$$\lim_{(\mu)} \int_a^{A-1/\mu} f(x) \phi(x) \, dx = \int_a^A f(x) \phi(x) \, dx,$$

the last integral being convergent because

$$|f(x) \phi(x)| < L\bar{\phi}(x).$$

And the right-hand side of (3) is equal to

$$\lim_{(\nu)} \int_a^A f(x) \left[\sum_0^\nu \phi_n(x)\right] dx = \sum_0^\infty \int_a^A f(x) \phi_n(x) dx.$$

Thus we find the equation

$$(4) \qquad \int_a^A f(x) \phi(x) \, dx = \sum_0^\infty \int_a^A f(x) \phi_n(x) \, dx,$$

which is Hardy's theorem.*

It is easy to modify the foregoing proof so as to obtain the result

$$(4') \qquad \int_a^\infty f(x) \phi \, dx = \sum_0^\infty \int_a^\infty f(x) \phi_n(x) \, dx$$

by postulating, in place of Hardy's condition (iii), the convergence of the integral

$$\int_a^\infty |f(x)| \bar{\phi}(x) \, dx.$$

2. From the method of proof used in § 1, it is clear that Hardy's condition (iii) is narrower than is necessary to ensure the truth of equation (4). For this condition is sufficient to ensure the existence of the double limit

$$(5) \qquad \lim_{(\mu, \nu)} \int_a^{A-1/\mu} f(x) \left[\sum_0^\nu \phi_n(x)\right] dx,$$

and to prove further that (5) is equal to either side of (4);

* Communicated to me by Mr. Hardy towards the end of 1904: earlier in that year I had obtained the result in the special case when all the functions $\phi_n(x)$ are *positive*, using the method given above to establish equation (2).

but it is well known* that the existence of (5) is not implied by the truth of (4), and therefore Hardy's test must be too narrow. An example is afforded by the series

$$(6)\qquad \phi(x) = \frac{\sin x}{x} + \frac{\sin 2x}{2^p x} + \frac{\sin 3x}{3^p x} + \dots \qquad (p > 1),$$

which is uniformly convergent if x is not less than 1; so that

$$(7)\qquad \int_1^\mu \phi(x)\,dx = \sum_1^\infty \int_1^\mu \frac{\sin nx}{n^p x}\,dx.$$

Now

$$\int_\mu^\infty \frac{\sin nx}{n^p x}\,dx = \left[-\frac{\cos nx}{n^{p+1} x}\right]_\mu^\infty - \int_\mu^\infty \frac{\cos nx}{n^{p+1} x^2}\,dx,$$

from which we see that

$$(8)\qquad \left|\int_\mu^\infty \frac{\sin nx}{n^p x}\,dx\right| < \frac{2}{\mu n^{p+1}}.$$

And therefore,† from (7) and (8),

$$\left|\int_1^\mu \phi(x)\,dx - \sum_1^\infty \int_1^\infty \frac{\sin nx}{n^p x}\,dx\right| < \frac{2}{\mu} \sum_1^\infty \frac{1}{n^{p+1}}.$$

Hence we have

$$(9)\qquad \int_1^\infty \phi(x)\,dx = \sum_1^\infty \int_1^\infty \frac{\sin nx}{n^p x}\,dx.$$

But the integral $\int_1^\infty \bar{\phi}(x)\,dx$ does not converge; for

$$|\phi_n(x)| > \frac{\sin^2 nx}{n^p x} = \frac{1 - \cos 2nx}{2n^p x}.$$

Thus $$\bar{\phi}(x) > \frac{1}{2x} \sum_1^\infty \frac{1}{n^p} - \tfrac{1}{2} \sum_1^\infty \frac{\cos 2nx}{n^p x},$$

and so

$$\int_1^\mu \bar{\phi}(x)\,dx > \tfrac{1}{2} \log \mu \sum_1^\infty \frac{1}{n^p} - \tfrac{1}{2} \sum_1^\infty \int_1^\mu \frac{\cos 2nx}{n^p x}\,dx.$$

* See for example Pringsheim's paper quoted above.

† Since $\left|\int_1^\infty \frac{\sin nx}{n^p x}\,dx\right| < \frac{2}{n^{p+1}}$, it is clear that the series $\sum_1^\infty \int_1^\infty \frac{\sin nx}{n^p x}\,dx$ is absolutely convergent.

But, just as in proving (8), we have

$$\left|\int_1^\mu \frac{\cos 2nx}{n^p x}\,dx\right| < \frac{1}{\mu n^{p+1}},$$

and therefore

$$(10) \qquad \int_1^\mu \overline{\phi}(x)\,dx > \tfrac{1}{2}\log\mu \sum_1^\infty \frac{1}{n^p} - \frac{1}{2\mu}\sum_1^\infty \frac{1}{n^{p+1}}.$$

From (10) it is plain that $\int_1^\infty \overline{\phi}(x)\,dx$ is divergent, in spite of the fact that term-by-term integration is permissible, as may be seen from equation (9).

The examples in § 4 would also be excluded by the test that $\int_0^1 \overline{\phi}(x)\,dx$ is to converge.

3. The following test can be used to justify the term-by-term integration of certain power-series which would be excluded under Hardy's test:—

If the power-series $\phi(x) = \Sigma a_n x^n$ has the circle $|x| = 1$ as its circle of convergence, and if $x = 1$ is a point at which the series does not converge, then the equation*

$$(11) \qquad \int_0^1 f(x)\,\phi(x)\,dx = \Sigma a_n \int_0^1 x^n f(x)\,dx$$

is valid, provided (i) *that $f(x)$ is continuous throughout* (0, 1) (ii) *that the series $\Sigma a_n \int_0^1 x^n f(x)\,dx$ is convergent, and* (iii) *that the integral $\int_0^1 \phi(x)\,dx$ is absolutely convergent.*

For, in virtue of condition (ii), Abel's theorem gives

$$(12) \qquad \Sigma a_n \int_0^1 x^n f(x)\,dx = \lim_{s=1} \Sigma a_n \int_0^1 (xs)^n f(x)\,dx \qquad 0 < s < 1,$$

and by the properties of power-series, $\Sigma a_n (xs)^n$ converges uniformly from $x = 0$ to $x = 1$, so that

$$\Sigma a_n \int_0^1 (xs)^n f(x)\,dx = \int_0^1 f(x)\,[\Sigma a_n (xs)^n]\,dx = \int_0^1 f(x)\,\phi(xs)\,dx.$$

* If the series were convergent at $x = 1$, it would converge uniformly from $x = 0$ to $x = 1$, by Abel's theorem. Thus equation (11) would follow from the ordinary rule.

From the consideration of a number of special series, it seems likely that when the series *diverges* at $x = 1$, the test above can only be applied if Hardy's test can be used. That is, fresh information is only afforded when Σa_n *oscillates*; but so far I have no proof of this conjecture.

From the last result, and from (12), we see that

$$(13) \qquad \Sigma a_n \int_0^1 x^n f(x)\,dx = \lim_{s=1} \int_0^1 f(x)\,\phi(xs)\,dx$$

$$= \lim_{s=1} \frac{1}{s} \int_0^s f\left(\frac{x}{s}\right) \phi(x)\,dx.$$

Now

$$(14) \quad \int_0^s f(x/s)\,\phi(x)\,dx - \int_0^1 f(x)\,\phi(x)\,dx$$

$$= \int_0^s [f(x/s) - f(x)]\,\phi(x)\,dx - \int_s^1 f(x)\,\phi(x)\,dx$$

Since $f(x)$ is continuous (and therefore *uniformly* continuous) from $x=0$ to $x=1$, a constant ρ can be found, corresponding to any positive number σ, such that

$$|f(x') - f(x)| < \sigma,$$

provided that $|x' - x| < \rho$. Hence

$$|f(x/s) - f(x)| < \sigma,$$

provided that $|x/s - x| < \rho$; and the last condition is satisfied as x ranges from 0 to s, provided that $1 - s < \rho$. Thus

$$(14') \quad \left| \int_0^s [f(x/s) - f(x)]\,\phi(x)\,dx \right| < \sigma \int_0^s |\phi(x)|\,dx < K\sigma,$$

if

$$1 - s < \rho,$$

where

$$K = \int_0^1 |\phi(x)|\,dx.$$

Again $$\left| \int_s^1 f(x)\,\phi(x)\,dx \right| < L \int_s^1 |\phi(x)|\,dx,$$

and a value ρ' can be found to make this last integral less than σ', provided that $1 - s < \rho'$, in virtue of condition (iii).

Thus

$$(14'') \qquad \left| \int_s^1 f(x)\,\phi(x)\,dx \right| < L\sigma', \quad \text{if } 1 - s < \rho'.$$

From (14), (14′), and (14″) it is evident that, given ϵ, we can find δ such that

$$\left|\int_0^s f(x/s)\,\phi(x)\,dx - \int_0^1 f(x)\,\phi(x)\,dx\right| < \epsilon, \qquad \text{if } 1-s<\delta.$$

Thus $$\lim_{s=1}\int_0^s f(x/t)\,\phi(x)\,dx = \int_0^1 f(x)\,\phi(x)\,dx,$$

and from this equation and (13), the truth of (11) is evident.

It is, however, sometimes difficult in practice to test the convergence of $\Sigma a_n \int_0^1 x^n f(x)\,dx$ without making further hypotheses as to $f(x)$; and in some cases it is convenient to adopt the following form of test:—

Suppose that $\lim_{n=\infty}(a_n/a_{n+1}) = \lambda$, *where* λ *is either* -1 *or a complex number such that* $|\lambda| = 1$*; *then it may happen that Hardy's test will apply to the series* $\sum_0^\infty (a_n - \lambda a_{n+1})\,x^{n+1}$, *and if so equation* (11) *is valid provided that* $\lim_{n=\infty} a_n \int_0^1 x^n f(x)\,dx = 0$.

For write

$$\psi(x) = \sum_0^\infty (a_n - \lambda a_{n+1})\,x^{n+1} - \lambda a_0 = (x-\lambda)\,\phi(x).$$

Then, by Hardy's test,

$$(15)\quad \int_0^1 \frac{\psi(x)f(x)}{x-\lambda}\,dx = \sum_0^\infty (a_n - \lambda a_{n+1})\int_0^1 \frac{x^{n+1}f(x)}{x-\lambda}\,dx - \lambda a_0\int_0^1 \frac{f(x)}{x-\lambda}\,dx,$$

because $1/(x-\lambda)$ is continuous from $x=0$ to 1.

In the series on the right-hand side of (15), we may collect together the terms multiplied by a_n, provided that

$$(16)\qquad \lim_{n=\infty} a_n \int_0^1 \frac{x^{n+1}f(x)}{x-\lambda}\,dx = 0.$$

---

* The case $\lambda = 1$ must be excluded, to ensure the continuity of $(x-\lambda)^{-1}$. Hardy's test here will postulate the convergence of

$$\int_0^1 \bar{\psi}(x)\,dx,$$

where $$\bar{\psi}(x) = \sum_0^\infty |a_n - \lambda a_{n+1}|\,x^{n+1}.$$

The equation (15) then reduces to (11); and, since $|x/(x-\lambda)|$ lies between finite limits as x ranges from 0 to 1, it follows that the condition (16) is certainly satisfied if

$$(16') \qquad \lim_{n=\infty} a_n \int_0^1 x^n f(x)\,dx = 0,$$

which again is satisfied if

$$(16'') \qquad \lim_{n=\infty} (a_n/n) = 0,$$

because of the continuity of $f(x)$.

4. To illustrate the tests in § 3, we may consider first the series

$$\phi(x) = 1 + \lambda x + \lambda^2 x^2 + \ldots, \quad \text{where } |\lambda| = 1.$$

Here λ takes the place of $1/\lambda$ in § 3; and $a_{n+1} - \lambda a_m = 0$.

Then if $f(x) = x^{p-1}$, it is evident that condition (i) is satisfied when the real part of p is greater than 1; and the integral $\int_0^1 \phi(x)\,dx$ is absolutely convergent if λ is not equal to 1. Further, the series

$$\Sigma\lambda^n \int_0^1 x^n f(x\,dx) = \Sigma \frac{\lambda^n}{n+p}$$

is known to converge; and therefore the equation

$$(17) \qquad \int_0^1 \frac{x^{p-1}}{1-\lambda x}\,dx = \sum_0^\infty \frac{\lambda^n}{n+p}$$

is valid.*

Again, consider the series

$$\phi(x) = (1+x)^{-\alpha} = 1 + \sum_1^\infty (-1)^n \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1.2\ldots n} x^n,$$

where β (the real part of α) satisfies $1 \leqq \beta < 2$. Here we find $\lambda = -1$ and

$$|a_n + a_{n+1}| = \left| \frac{(\alpha-1)\,\alpha\ldots(\alpha+n-1)}{1.2\ldots(n+1)} \right|,$$

---

* As a matter of fact (17) is established more quickly by means of the identity

$$\sum_0^{\nu-1} \frac{\lambda^n}{n+p} = \int_0^1 \frac{x^{p-1}}{1-\lambda x}\,dx - \int_0^1 \frac{\lambda^\nu x^{\nu+p-1}}{1-\lambda x}\,dx;$$

and it will be seen that the real part of p need only be positive.

which is the absolute value of the coefficient of x^{n+1} in $(1-x)^{-(a-1)}$. Hence, as proved by Hardy (see § 1 of his paper), the series $\sum\limits_0^\infty |a_n + a_{n+1}| x^{n+1}$ satisfies Hardy's test; thus the second form of test given in § 3 establishes the equation

$$(18)\quad \int_0^1 \frac{f(x)}{(1+x)^a} dx = \int_0^1 f(x)\, dx$$
$$+ \sum_1^\infty (-1)^n \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{1.2\ldots n} \int_0^1 x^n f(x)\, dx,$$

because $\lim\limits_{n=\infty} a_n/n = 0$ in virtue of the known results as to the binomial series.

5. It may not be superfluous to remark that Abel's theorem can be applied in other ways, so as to extend the range of values for which term-by-term integration is permissible. For example, Hardy's test gives (compare § 5 of his paper)

$$(19)\qquad \int_0^\infty e^{-ax} \frac{\sin bx}{x} dx = \sum_0^\infty \frac{(-1)^n}{2n+1} \left(\frac{b}{a}\right)^{2n+1},$$

only if $-a < b < a$; but Abel's theorem shows that this equation is valid even for $b = a$.

Abel's identity leads very easily also to the following test:—

The equation $\int_0^A (\sum\limits_0^\infty v_n \phi_n)\, dx = \sum\limits_0^\infty \int_0^A (v_n \phi_n)\, dx$ *is valid, provided* (i) *that* v_n *and* ϕ_n *are continuous throughout* (a, A); (ii) *that* $|V_\nu| = |\sum\limits_0^\nu v_n|$ *is less than a constant* L, *for the whole of* (a, A); (iii) *that* $\lim\limits_{n=\infty} \int_a^A |\phi_n(x)|\, dx = 0$ *and* $\lim\limits_{n=\infty} \phi_n(x) = 0$ *throughout* $(a, A-\epsilon)$; *and* (iv) *that* $\psi(x) = \sum\limits_0^\infty |\psi_n(x)|$ *converges uniformly throughout* $(a, A-\epsilon)$, *and* $\int_a^A \psi(x)\, dx$ *is convergent where* $\psi_n(x) = \phi_n(x) - \phi_{n+1}(x)$.

But this test does not lend itself very readily to practical work, and all cases which I have found to be covered by it are also included in the tests already explained; I do not, therefore, stay to write out the proof.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy*, Trinity College, Cambridge.

XIX.

On Abel's Lemma and the second Theorem of the Mean.

§ 1. A VERY large number of theorems in the Theory of Series and the Integral Calculus are immediate or almost immediate consequences of the very elementary identity (generally known as *Abel's Lemma*) expressed by the equation

$$(1)\quad \sum_m^n a_\nu f_\nu = \sum_m^{n-1} (a_m + a_{m+1} + \ldots + a_\nu)(f_\nu - f_{\nu+1}) + (a_m + \ldots + a_n) f_n.$$

Among the most important of these theorems is that usually described as "the second Theorem of the Mean," the theorem which asserts that if*

(i) $f(x)$ is a function of x which steadily increases (or decreases) as x varies from a to A,

(ii) $\int_a^A \phi(x)\,dx$ is convergent, then

$$\int_a^A f(x)\,\phi(x)\,dx = f(a+0)\int_a^\xi \phi(x)\,dx + f(A-0)\int_\xi^A \phi(x)\,dx.$$

My object in this note is to call attention to an identity which closely resembles Abel's, and of which substantially the same applications may be made, but which has the advantage over Abel's of greater symmetry and of immediately suggesting the second Mean Value Theorem by its form. In fact, the latter theorem is strictly a limiting case of the identity which I shall establish, just as the formula for integration by parts is a limiting case of the ordinary form of Abel's Lemma.†

§ 2. The identity in question is

$$\begin{aligned} a_m f_m + a_{m+1} f_{m+1} + \ldots + a_n f_n \qquad & \\ &+ \frac{f_m - f_{m+1}}{f_m - f_n}\{f_m a_m + f_n(a_{m+1} + a_{m+2} + \ldots + a_n)\} \\ &+ \frac{f_{m+1} - f_{m+2}}{f_m - f_n}\{f_m(a_m + a_{m+1}) + f_n(a_{m+2} + \ldots + a_n)\} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \frac{f_{n-1} - f_n}{f_m - f_n}\{f_m(a_m + \ldots + a_{n-1}) + f_n a_n\}. \end{aligned}$$

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* Jordan, *Cours d'Analyse*, t. II., p. 220.

† v. Note 1, *Messenger*, Vol. XXX., p. 185.

That this is an identity may be verified at once; in fact, the coefficient of a_r is

$$\frac{1}{f_m-f_n}[f_n(f_m-f_{m+1}+f_{m+1}-f_{m+2}+\ldots+f_{r-1}-f_r)$$
$$+f_m(f_r-f_{r+1}+f_{r+1}-f_{r+2}+\ldots+f_{n-1}-f_n)]$$
$$=\frac{1}{f_m-f_n}\{f_n(f_m-f_r)+f_m(f_r-f_n)\}=f_r.$$

§ 3. From this identity we may immediately deduce the convergence theorems of Abel and Dirichlet, viz., that $\Sigma a_n f_n$ is convergent if either

(i) Σa_n is convergent and $f_n \geqq f_{n+1}$;

(ii) Σa_n oscillates between finite limits of indetermination $f_n \geqq f_{n+1}$ and $\lim f_n = 0$.

But the identity of § 2 has no advantages over Abel's as a step towards the proof of these theorems.

§ 4. It enables us, however, to give a very simple and direct proof of the second Mean Value Theorem. We assume that (i) $f(x)$ steadily decreases from $x=a$ to $x=A$, and (ii) $\phi(x)$ is finite and integrable. Let us divide (a, A) into n small intervals $\delta x_0, \delta x_1, \ldots \delta x_{n-1}$. Then $f(x)\phi(x)$ is integrable and

$$\int_a^A f(x)\phi(x)\,dx = \lim_{n=\infty} \sum_{\nu=0}^{n-1} f_\nu \phi_\nu \delta x_\nu,$$

where f_ν and ϕ_ν are the values of f and ϕ at any point inside the interval δx_ν.

The finite sum on the right-hand side is, by the lemma, a quantity lying between the greatest and least values of

$$f_0(\phi_0\delta x_0+\phi_1\delta x_1+\ldots+\phi_{r-1}\delta x_{r-1})+f_{n-1}(\phi_r\delta x_r+\ldots+\phi_{n-1}\delta x_{n-1})$$

for $r=1, 2, \ldots, n-1$.

Let $$I_{\mu,\nu}=\int\phi dx,$$

extended over the aggregate of intervals $\delta x_\mu, \delta x_{\mu+1}, \ldots, \delta x_\nu$. Then we can choose N so that for $n>N$ and for all values of μ and ν

$$|\phi_\mu\delta x_\mu+\ldots+\phi_\nu\delta x_\nu - I_{\mu,\nu}|<\epsilon,$$

however small be ϵ. For the left-hand side is not greater than

$$\sum_{s=\mu}^{\nu} O_s\delta x_s \leqq \sum_{s=0}^{n-1} O_s\delta x_s,$$

where O_s is the oscillation of ϕ in δx_s, and the limit of the right-hand side is zero.

Hence, if any small positive quantity η is given, we can choose N so that for $n > N$

(i) $|\{f_0 - f(a+0)\} I_{0,r-1}| < \eta \qquad (r = 1, 2, \ldots, n)$,

(ii) $|\{f_{n-1} - f(A-0)\} I_{r,n}| < \eta \qquad (r = 1, 2, \ldots, n)$,

and (iii) the integral

$$I = \int_a^A f(x)\,\phi(x)\,dx$$

lies between $m - \eta$ and $M + \eta$, where m and M are the least and greatest of the quantities

$$f_0 I_{0,r-1} + f_{n-1} I_{r,n-1}:$$

and so
$$m_1 - 3\eta < I < M_1 + 3\eta,$$

where m_1 and M_1 are the least and greatest values of

$$f(a+0)\,I_{0,r-1} + f(A-0)\,I_{r,n-1}.$$

A fortiori,

$$m_1' - 3\eta < I < M_1' + 3\eta$$

if m_1' and M_1' are the least and greatest values of

$$f(a+0)\int_a^\xi \phi\,dx + f(A-0)\int_\xi^A \phi\,dx.$$

As m_1' and M_1' do not depend upon n, these inequalities involve

$$m_1' \leqq I \leqq M_1';$$

and the theorem follows.

§ 5. If we had taken $f_0, \phi_0, f_1, \phi_1, \ldots$ to be the values of f and ϕ *at the lower ends* of $\delta x_0, \delta x_1, \ldots$, we should have been led to the formula

$$I = f(a)\int_a^\xi \phi(x)\,dx + f(A)\int_\xi^A \phi\,dx.$$

Since
$$f(a) \geqq f(a+0) \geqq f(A-0) \geqq f(A),$$

this is perfectly consistent with the former result; in fact, if H and K are any quantities such that

$$H \geqq f(x) \geqq K,$$

we may write

$$I = H\int_a^\xi \phi(x)\,dx + K\int_\xi^A \phi(x)\,dx.$$

This is an immediate consequence of the following lemma:

If (i) $$\alpha \geqq \alpha' \geqq \beta' \geqq \beta,$$

(ii) $F(x)$ *is a continuous function of* x *which is equal to* 0 *for* $x = a$ *and to* C *for* $x = A$, *then the limits of variation of*

$$\alpha F + \beta (C - F)$$

include between them those of

$$\alpha' F + \beta' (C - F),$$

and so every value of the latter function is a value of the former.

To prove this we have only to observe that each function is a maximum or minimum when F is so, since $\alpha - \beta \geqq 0$, $\alpha' - \beta' \geqq 0$. Suppose, *e.g.*, $C > 0$. Then the maximum of F is $\geqq C$ and the minimum $\leqq 0$. But

$$(\alpha - \alpha') F + (\beta - \beta') (C - F) = \{\alpha - \beta - (\alpha' - \beta')\} F + (\beta - \beta') C$$

is $\geqq 0$ for $F = C$ and $\leqq 0$ for $F = 0$, and it never decreases as F increases. Thus it is $\geqq 0$ when F is greatest and $\leqq 0$ when F is least: which proves the lemma.

THE MULTIPLICATION OF AN INFINITY OF ORDINAL TYPES.

By *Philip E. B. Jourdain.*

THE following method of defining the product of an infinity (the definition of Cantor* being framed for merely a *finite* number) of ordinal types originated in reading a method given by Hausdorff† of defining that type which should be—in analogy with Cantor's *Belegungsmenge*‡—represented by the

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* *Math. Ann.*, Bd. XLVI. (1895), pp. 502—503.

† "Der Potenzbegriff in der Mengenlehre," *Jahresber. der Deutsch. Math. Ver.*, Bd. XIII. (1904), pp. 569—571.

‡ *Loc. cit.*, pp. 486—487. Cantor's definition is merely for *cardinal* numbers [cf. also Schoenflies, "Die Entwickelung der Lehre von den Punktmannigfaltigkeiten," Leipzig, 1900, pp. 8–9; Whitehead, *Amer. Journ. of Math.*, Vol. XXIV. (1902), pp. 369, 385, 388]. Cantor has used the exponential *notation* for certain ordinal numbers (see, *e.g.*, *Math. Ann.*, Bd. XLIX. (1897), pp. 231 *seq.*), and this notation is convenient and has generally been adopted, but there is no analogy between a series of (for example) the type denoted by Cantor ω^ω, and a series whose terms are the various coverings of a series of type ω with a series of type ω. For the cardinal number of any of the latter series (whatever the particular order may be) is $\aleph_0^{\aleph_0}$, while the cardinal number of a series of type ω^ω is $\aleph_0$, and

$$\aleph_0^{\aleph_0} > \aleph_0.$$

Throughout the present paper we shall only consider series which *are* ordered "Belegungsmengen," so there is no fear of ambiguity when we use the ordinary notation μ^ν. If Cantor's conception of α^β (where α and β are ordinal numbers) occurs, it may be useful to mark the distinction of the former conception by the notation $(\mu)^{(\nu)}$.

notation μ^α, where μ is a type and α an ordinal number. It was a question of generalising, as far as possible, Cantor's conception.

I.

Where μ and ν are the ordinal types of series M and N, Cantor defined the product $\mu.\nu$ as follows: *

In N, in the place of every element n, let a series M_n of type μ be substituted; the type of the resulting series is defined to be $\mu.\nu$.

The "Verbindungsmenge" or "multiplicative class" of two aggregates M and N, which serves to define the product $\mathfrak{m}.\mathfrak{n}$ of their *cardinal* numbers, is defined as the cardinal number of the aggregate of all couples (m, n), where m is an element of M and n of N.† It is evident that a similar‡ aggregate is obtained by replacing each element of N by an aggregate of cardinal number $\mathfrak{m}$; also we may, if M and N are simply ordered aggregates of types μ and ν respectively, define $\mu.\nu$ as the aggregate of couples (m, n) whose order is determined as follows:

If (m, n) and (m', n') are any two couples, and $n > n'$,§ then $(m, n) > (m', n')$ whatever the relation of m to m' may be; also if $n < n'$, then $(m, n) < (m', n')$. But if $n = n'$, then $(m, n) \gtrless (m', n')$ according as $m \gtrless m'$.

We can easily generalise this to the case of the multiplication of any finite number of types $\alpha, \beta, \ldots, \mu$ belonging to series $A, B, \ldots, M$. Thus, $\alpha.\beta \ldots \mu$ is the type of the series of elements $(a, b, \ldots, m)$, where any two elements have the same rank as the last corresponding terms in the complexes which differ. That is to say, $(a, b, \ldots, m)$ and $(a', b', \ldots, m')$ have the same rank as m and m', if m and m' differ; but if $m = m'$ and k, k' are the last corresponding terms in $(a, b, \ldots, m)$, $(a', b', \ldots, m')$ to differ, $(a, b, \ldots, m)$ and $(a', b', \ldots, m')$ have the same respective rank as k and k'.

II.

Obviously, this can be further generalised to the case where the type of the elements $(a, b, \ldots, m)$ is an *inverse ordinal number*, that is to say, each element has a last term,

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* *Math. Ann.*, Bd. XLVI. (1895), pp. 502—503.

† Cantor, *ibid.*, p. 485.

‡ Cantor's "äquivalent."

§ Here $n > n'$ means "n follows n'," and $n < n'$ means "n precedes n'."

and each series of terms contained in the element has a last term, but the element need not have a first term. Suppose, then, that each element is of type $^*\zeta$, where ζ is an ordinal number, we can define the product of a series of type $^*\zeta$ of types μ_γ, which may be represented

$$(1) \qquad \prod_{^*\zeta < \gamma \leqq {^*1}} \mu_\gamma$$

When all the types μ_γ are equal to μ, (1) reduces to μ exponentiated by $^*\zeta$, or

$$\mu^{^*\zeta}.$$

III.

We can now see that Hausdorff's conception of an ordered covering-aggregate (Belegungsmenge) is not a natural extension of Cantor's conception of an ordered multiplicative-class (Verbindungsmenge) of a finite number of series. For Hausdorff's definition of μ (any type) raised to the power α (any ordinal number) is the type of the series composed of all terms

$$(x_0, x_1, x_2, \ldots, x_\omega, x_{\omega+1}, \ldots) \ldots\ldots\ldots\ldots\ldots (2),$$

each term (2) being of type α, and each x in (2) being a term of the series of type μ; and any two such terms as (2) have the rank of the two *first* corresponding sub-terms x which differ. We have seen that the natural extension of Cantor's definition is to make the rank in question decided by the two *last* corresponding terms x which differ, and consequently to frame the definition for any *inverse* ordinal number $^*\alpha$, instead of α; and Hausdorff, indeed, seems to recognise this afterwards by remarking that his μ^α has "the formal properties of $\mu^{*\alpha}$."

IV.

Exactly as the conception of the product of an infinite class of cardinal numbers involves the difficulty that, to make it valid, we must prove, or assume as an axiom, that the multiplicative class contains at least one term (is not the "null-class"), so with the product of a series of type

$$^*\zeta$$

of ordered types μ_γ, we meet what we may call the "necessity

of the multiplicative axiom." This axiom may be described, and its importance indicated, as follows.

If k is a class of classes (none of which latter are null), what we may call the 'multiplicative class of k' is the class, each of whose terms is a class formed by choosing one, and only one, term from each of the classes belonging to k. If the classes belonging to k are such that no two of them have any term in common (k is "a class of mutually exclusive classes"), the cardinal number of terms in the multiplicative class of k is said to be the *product* of all the cardinal numbers of the various classes composing k.* Now it would probably be considered evident that the multiplicative class is not null, and this seems to have been the opinion of Whitehead when writing his paper. But it appears to have been Russell who drew attention (though nothing on this subject has hitherto been published, even in Russell's book) to the fact that it is, in general, an assumption to say that a multiplicative class is not null ('exists').† To prove this existence-theorem we require to be able to give a law which should pick out one term out of each of the classes composing K; in other words, there must be a term in each of these classes which is what we may call *special* (praesignum, ausgezeichnete). It is not sufficient to know (what seems true) that, if x is a term of the class a, there is at least one relation R of x to a that no other element of a but x has to a; but, if a is a term of k, one must *determine* (by some law) *one* such x for *every* such a.

[*Note added during the correction of proofs, March* 26, 1906: The difficulty of the multiplicative, and the analogous Zermelo's, axiom has been dealt with lately by Russell (*Proc. Lond. Math. Soc.* (2), Vol. IV., 1906, pp. 47–53) and Hardy (*ibid*, pp. 14–17). I propose to return to the subject in a communication on the Heine-Borel theorem.]

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* Cf. Whitehead, *loc. cit.*, pp. 369, 383, 385; Russell, "The Principles of Mathematics," Vol. I., Cambridge, 1903, pp. 119, 308–309.

† If we wish to say 'x is a member of the class a,' it is a necessary hypothesis to know that a 'exists' (that it has at least one member, or is not null) (cf. Whitehead, *loc. cit.*, p. 373).

The Manor House,
Broadwindsor,
Dorset,
March 10*th*, 1906.

EVIDENCE OF GOLDBACH'S THEOREM.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., London.

[The author's acknowledgments are due to Mr. H. J. Woodall, A.R.C.Sc, for reading the proof-sheets of this Paper.]

1. *Goldbach's Theorem.* The important Theorem that—

Every even number (E) is the sum of two primes ($p + q$).........(1),

has remained to the present day without strict proof. The enunciation of this as a *probable* Theorem has been traced back to a letter* from Christian Goldbach to L. Euler in 1742; and Euler, in reply, expresses his belief in its truth. The theorem was also enunciated† by Ed. Waring, but at a later‡ date (1782?).

2. *Present Evidence.* The evidence of this Theorem depends at present on *numerical verification.* The details of this verification, as yet published, are contained in the following Works.

1. Cantor, G. *Vérification jusqu' à* 1000 *du Théorème empirique de Goldbach;* pub. in Report of the *Assocn. Française pour l'Avancement des Sciences,* for 1894. This pamphlet of 18 pages contains only a few lines of explanatory text. The Table, pp. 1—18, gives the *smaller prime* (x) for *every partition* of every *even* number ($2N$) from 2 to 1000 into a *sum of two primes* (x, y); and also the number (n) of such partitions for each such even number ($2N$)

2. Aubry, V. See *L'Intermédiaire des Mathematiciens,* t. iii., 1896, p. 75, and t. iv., 1897, p 60; Mention is made of a Table of solutions of $E=p+q$ from 1002 to 2000, prepared by M. V. Aubry (not yet published).

3 Haussner, R. *Tafeln für das Goldbach'sche Gesetz.*, pub. in *Abh. der Kaiserl. Leop. Carol. Deutschen Akad. der Naturforscher*, Bd. i, Halle, 1899. This Work, of 214 quarto pages, gives an explanatory text with a theoretical discussion, pp. 5 to 21, followed by four Tables.

Tab. I., pp. 25—178, gives the *smaller prime* (x) for *every partition* of every even number ($2n$) from 2 to 3000 into a *sum of two primes* (x, y), and also the *number* (ν) of such partitions for each such even number ($2n$).

Tab. II, pp. 181—191, gives the *number* (ν) of such partitions ($2n = x + y$) for every *even number* ($2n$) from 2 to 5000.

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* See *Correspondance mathém. et phys. de célèbres géom. du* 18ème *siècle*, by P. H. Fuss, Vol. i., St. Petersburgh, 1843. Letters Nos. 43, 44; pp. 125—136. These letters are quoted, with pretty full extracts, in Herr R. Haussner's Work (see Art. 2, below), pp. 18, 19.

† see E. Waring's *Meditationes Algebraicae*, Cambridge, 1782, p. 379.

‡ Ed. Lucas ascribes the Theorem to Waring (see Lucas's *Théorie des Nombres*, Vol. i., Paris, 1891, p. 353), as enunciated in his *Meditationes Analyticae* (which was published in 1785): but Herr Haussner reports this reference to be *incorrect* (see p. 19 of his Work quoted below) and states that the reference should be to Waring's *Meditationes Algebraicae* of 1770 (this latter Work is however dated 1782). Neither Lucas nor Haussner quote the page containing the theorem; so that it is difficult to verify their statements: as neither of Waring's works is indexed.

Tab. III., pp. 195—210, gives the values of $P(\sigma)$, and $\xi(\sigma)$ for all *odd* numbers $\sigma = 2\rho + 1$ from 1 to 4999, where

$P(\sigma)$ = the number of odd primes from 1 to σ (including σ when prime).

$$\xi(\sigma) = P(2\rho + 1) - 2P(2\rho - 1) + P(2\rho - 3).$$

The author also states (p. 10) that he has extended the (simple) verification of the Theorem to 10000 without finding any exception.

4. Ripert, L. See *L'Intermédiaire des Mathématiciens*, t. x., 1903, pp. 66, 166; The number (ν) of solutions of $E = p + q$ is given for a few specially selected higher values of $E > 3000$ up to 50816, with details (p, q) in some cases.

2*a*. *Summary of evidence.* It will be seen that this evidence consists at present solely of numerical verification.

1.	*H. Cantor.*	Up to 1000.	Full detail; all published.
2.	*M. Aubrey.*	1002 to 2000.	Full detail; not published (?).
3.	*H. Haussner.*	Up to 3000.	Full detail; all published.
	,,	3000 to 5000.	Full detail; only values of ν published.
	,,	5000 to 10000.	Simple verification; not published.
4.	*M. Ripert.*	Up to 50816.	Full detail of a few special cases only; Value of ν given for a few special cases.

Thus the numerical verification extends at present continuously only to 10000, and for a few selected cases to 50816.

3. *Possible failure.* In a Paper in the *Nouv. Ann. de Mathém.*, 1879, p. 356, M. Lionnet attempts to show that there is some *probability* of failure of the Theorem for *very high numbers.* The argument is based chiefly on the gradual increase of the ratio of the number of composites to the number of primes within a given range as the numbers increase. The Argument does not seem very convincing (to the present author).

4. *New numerical evidence.* It seemed to the author that it would be interesting to extend the numerical verification among certain classes of numbers which might be looked on as *critical numbers*, or as a sort of *Test-numbers* of the Theorem, and to push it to high limits among those numbers. The Test-numbers chosen are of the three following classes which may be described as

i. *Hyper-Even*;	ii. *Semi-Hyper-Even*;	iii. *Barely Even.*
2^n, $2^n.\omega$;	$(4\omega)^n$, $2.(2\omega)^n$, $(2\omega)^n$, $2^n.(2^n \mp 1)$;	$2.\omega^n$, $2.(2^n \mp \omega)$.

where ω denotes a small *odd* number.

The numbers actually chosen are all the following

i. *Hyper-Even*; 2^n, 3.2^n, 5.2^n, 7.2^n, 9.2^n, 11.2^n.

ii. *Semi-Hyper-Even*; $(4.3)^n$, $(4.5)^n$; 2.10^n; $(2.3)^n$, $(2.5)^n$, $(2.7)^n$, $(2.9)^n$, $(2.11)^n$; $2^n(2^n \mp 1)$.

iii. *Barely Even*; 2.3^n, 2.5^n, 2.7^n, 2.11^n; $2(2^n \mp 11)$, $2(2^n \mp 9)$, $2(2^n \mp 7)$, $2(2^n \mp 5)$, $2(2^n \mp 3)$, $2(2^n \mp 1)$.

The following notation is used—

E denotes *even number* $= p + q$ (if possible)........................(1),

p, q, denote primes, with $p \not> q$.

$m = \frac{1}{2}(q - p)$; so that $p = \frac{1}{2}E - m$, $q = \frac{1}{2}E + m$, [m may $= 0$]...(1a),

ν denotes the number of partitions of E into $E = p + q$.

The numerical verification of the Theorem is effected by finding values of p, q to satisfy the theorem for each *selected* value of E; and it evidently suffices to record *one* of the three quantities p, q, m; as any one (when known), gives the rest. The finding of any one partition (p, q) for each selected E would of itself be a sufficient verification for that number E: but there are two particular partitions, of opposite character, which seem of such special interest that it has been thought worth while to determine *both*; these are

1°. $(q - p)$ a *maximum*; this involves—

p a *minimum*, ($=1$, 2, 3, &c., or always *small* compared with E).

q a *maximum*, (*i.e.* a little less than E).

2°. $(q - p)$ a *minimum*; this involves—

m a *minimum*, ($=0$, 1, 2, 3, &c., or always *small* compared with E).

p, q each differing little from $\frac{1}{2}E$.

and, for tabular record it suffices to show one quantity only, viz.

Case 1°. Only p is shown; *Case* 2°. Only m is shown.

4*a*. *Tab. I., II.* These Tables give the result (*i.e.* p or m) required for each of the two special partitions (1°, 2° above) for each of the Test-Numbers (E) above detailed.

Test-number form. The "form" of each Test-number is shown in the left column.

Test-number (E). The particular Test-number is defined by the value of the exponent (n) at head of each small column. Two lines are allotted to each "form" of E.

Upper or p-Line. This shows the *minimum* p (of partition 1°); except that when the minimum $p = 1$, the next greater value of p is also* given.

Lower or m-Line. This shows the *minimum* m (of partition 2°).

* This second value has been given to meet a possible objection to reckoning 1 as a *prime*; most of the previous writers (*e.g.* Messrs. Cantor, Aubry, Stäckel, Haussner, Ripert) admit 1 as a *prime*.

The two partitions in question have been effected in these two Tables for every value of n from $n = 1$ up to the highest limit found practicable (in some cases $E > 200$ million).* Success in this requires the knowledge of *all* the High Primes within a short range just below each value of E (for Case 1°), or of $\frac{1}{2}E$ (for Case 2°), and is therefore ordinarily limited to E or $\frac{1}{2}E \not> $ about 9* million. The limit (E max.) actually attained in each partition is shown below

	Form of $E = 2^n$,	$2(2^n \mp \omega)$;	2.3^n;	10^n,	2.10^n;	Other forms.
1°.	E max. $= 2^{26}$,	$2(2^{25} \mp \omega)$;	2.3^{13};	10^9,	2.10^7;	$\not>$ 9 million.
2°.	E max. $= 2^{27}$,	$2(2^{26} \mp \omega)$;	2.3^{15};	10^8,	2.10^8;	$\not>$ 18 million.

Thus, in certain special forms (those depending on† 2^n, 3^n, 10^n) the high limit of $E \not> 100$ million, and in some cases 200 million, is here reached.

4*b*. *Tab. III.* In those cases in which the more interesting solutions (*i.e.* p, m both minimum) could not be carried beyond 9 million, a supplementary Table (Tab. III.) has been prepared giving *one* solution of each of the Test-numbers (E) carried (in the special case of 2^n) up to 2^{28}, and in all other cases up to $E \not> 100$ million.

In this Table, also, only the value of p is shown, *viz.* the lowest value of p obtainable with the stock‡ of High Primes available to the author.

4*c*. *Summary of results.* As the result of Tab. I., II., III.,§ Goldbach's Theorem is hereby verified for each of the Test-forms up to the high limit of $E \not> 200$ million.

5. *Number of partitions* (ν). A *simple* formula for the number (ν) of partition of $E = p + q$ is a desideratum. As, however, ν depends on the *relative distribution* of primes in the two regions 1 to $\frac{1}{2}E$, $\frac{1}{2}E$ to E, a simple formula is perhaps not to be expected.

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* Being the limit of the existing large Factor-Tables, with continuation to 9001020 in the second Paper quoted in the Note† below.

† This has been rendered possible by the aid of certain Factorisation-Tables of $(2^n \mp R)$, $(3^n \mp R)$, $(10^n \mp R)$ contained in three Papers on *Determination of Successive High Primes*, by the present author and Mr. H. J. Woodall jointly, pub. in *Messenger of Mathematics*, Vol. XXXI., 1901-2, pp. 165–176; Vol. XXXIV., 1904-5, pp. 72–89, and pp. 184–192.

‡ This has been rendered possible by a MS. Table of High Primes compiled by the present author, containing several thousand High Primes (*i.e.* primes > 9 million); this Table is only in MS., being in fact still in progress.

§ These Tables were prepared by the author, and have been checked *throughout* by an assistant (Miss A. Woodward).

6. *Sylvester's formula.* The late Prof. Sylvester seems to have arrived at some sort* of expression for ν, but the actual expression appears not to have been published; and, from the slight forecast of it that is* available, it would seem to have been a pretty complicated one, and to have been of form quite different to the one described below (Art. 7).

7. *Haussner's formula.* Herr Haussner gives† what he states to be an *accurate* expression for the number (ν) of partitions of the even number $E = 2n = p + q$,

$$\nu\ddagger = \tfrac{1}{2} \cdot \Sigma_{\rho=0}^{\rho=n-1} P(2n-2\rho-1) \cdot \xi(2\rho+1) + \tfrac{1}{2}j \ldots (2),$$

where $P(\sigma)$ = the number of *odd* primes from 1 to σ (including σ when prime)

$$\xi(2\rho+1) = P(2\rho+1) - 2P(2\rho-1) + P(2\rho-3) \ldots (3),$$

$$j = 1, \text{ when } n \text{ is } prime;\ j = 0, \text{ when } n \text{ is composite} \ldots (4).$$

The formula really includes *only*§ the solutions of $E = p + q$ in *odd* primes, so that the particular solution $E = 4 = 2 + 2$ is§ *excluded.*

The above formula, as it stands, is of little practical use for the actual calculation of the value of ν; as the summation would seem to involve computing every term separately, and then adding (a most tedious affair).

7*a*. *Reduction of Haussner's formula.* The original formula can be reduced to a form more convenient for calculation.

The apparently complicated factor $\xi(2\rho+1)$, defined by (3), which affects every term, is easily seen to have *only three values*, viz.

$$\begin{aligned}\xi(2\rho+1) &= +1, \text{ when } (2\rho+1) \text{ is } prime, \text{ and } (2\rho-1)\ composite \ldots\ldots (3a),\\ &= -1, \text{ when } (2\rho+1) \text{ is } composite, \text{ and } (2\rho-1)\ prime. \ldots\ldots (3b),\\ &= 0, \text{ when } (2\rho \pm 1) \text{ are } both\ prime, \text{ or } both\ composite \ldots\ldots (3c).\end{aligned}$$

It is also easy to see—

1. The cases of $\xi(2\rho+1) = 0$ are far the most numerous, amounting to

* See *Proc. Lond. Math. Soc.*, Vol. IV., 1871, p. 6.

† See p. 17 of his Memoir quoted; the notation has been here altered (to suit this Paper).

‡ The lower limit is printed $\rho = 1$ in the original; but this appears to be a misprint for $\rho = 0$, as here printed.

§ This is not stated in the original; but may be inferred from the construction of the formula.

about 2/3 of the whole number, so that the corresponding zero terms, $P(2n-2\rho-1).\xi(2\rho+1)=0$, might with advantage be *omitted* from the formula for ν.

2. When $\rho=0, 1, 2, 3$, &c., ..., in succession, the finite values of ξ (*i.e.* $\xi=\pm 1$) are $+$ and $-$ alternately (separated, it may be, by one or more *zero* values of ξ), beginning with $+1$; thus

$$\left.\begin{array}{l} \text{When } \rho=0,\ 9,\ 11,\ 15,\ 17,\ 21,\ 23,\ 25, \&c. \\ \xi(2\rho+1)=+1, -1, +1, -1, +1, -1, +1, -1, \&c. \end{array}\right\} \begin{array}{l}\text{The omitted values of } \rho \\ \text{all give } \xi(2\rho+1)=0.\end{array}$$

3. As every factor $P(2n-2\rho-1)$ in ν is necessarily $+$, the series consists (after omitting the zero terms) of alternate $+$ and $-$ terms.

Next, pairing every $+$ term with its following $-$ term, and using the *abbreviation*

$$D_r = P(E-x_r) - P(E-y_r) \ldots\ldots\ldots\ldots\ldots (5),$$

the expression for ν may now be written in the reduced form

$$\nu=\tfrac{1}{2}\Sigma_r D_r + \tfrac{1}{2}j, \text{ [the summation extending to } x_r \,\&\, y_r \not> E]\ldots(6),$$

where $x_r=$ every *odd prime* from 1 to $\not> E$, such that (x_r-2) is *composite.*

$y_r=$ every *odd composite* from 9 to $\not> E$, such that (y_r-2) is *prime.*

so that x_r, y_r have the following values (to Argument r),

$r=1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9, 10, 11, 12, 13, 14, 15, 16,\ 17,\ 18,\ 19,\ 20$,&c.

$x=1, 11, 17, 23, 29, 37, 41, 47, 53, 59, 67, 71, 79, 83, 89, 97, 101, 107, 113, 127$,&c.

$y=9, 15, 21, 25, 33, 39, 45, 49, 55, 63, 69, 75, 81, 85, 91, 99, 105, 111, 115, 129$,&c.

Thus:—

The series of x_r includes the *whole of the primes* from 1 to E, excepting the *larger* prime (q) of every pair of primes whose difference

$= 2$, (*e.g.* $q-2$, q).....(7),

and, $y_r = x_r + 2$, 4, 8, by following Rule—

$y_1=9$, with $x_1=1$; $y_r=x_r+2$ (if *composite*); $y_r=x_r+4$ (if *prime*); $[r>1]$...(7*a*).

Now, let X, Y be the greatest values of x, $y \not> E$,

,, ν_x, ν_y, ν_d be the number of terms in the x, y, D series, respectively,

Then X, Y, ν_x, ν_y, ν_d are connected as follows:

If $E>x_m$, but $<y_m$; then $X=x_m$, $Y=y_{m-1}$; $\nu_x=m$, $\nu_y=m-1$, $\nu_d=m$...(8*a*).

If $E>y_m$, but $<x_{m+1}$; then $X=x_m$, $Y=y_m$; $\nu_x=\nu_y=\nu_d=m$ ....(8*b*).

The reduced formula (6) is much easier to compute than the original (2), in that

1°. It contains only $\frac{1}{3}$ of the number of terms $P(E-x_r)$ and $P(E-y_r)$ of the original.

2°. It suffices to compute the *differences* $D_r=P(E-x_r)-P(E-y_r)$, instead of computing the $P(E-x_r)$, $P(E-y_r)$ themselves, a much easier matter, being done by simply counting the number of primes in the *small interval* (y_r-x_r) which begins with $(E-x_r)$ and ends just before $(E-y_r)$.

As the interval $(y_r - x_r) =$ only 2 or 4 when $r > 1$, the number of primes (D_r) to be counted for each value of r is *very small*, never > 3; in fact

1°. $D_1 = 0, 1, 2, 3$, when there are 0, 1, 2, 3 primes (and no more) among the four numbers $E-1$, $E-3$, $E-5$, $E-7$........................(9*a*).

2°. When $y_r = x_r + 2$, and $r > 1$, then

$D_r = 0$, if $(E - x_r)$ is composite; $D_r = 1$, (if $E - x_r$) is *prime*......(9*b*).

3°. When $y_r = x_r + 4$, and $r > 1$, then

$D_r = 0, 1, 2$, when there are 0, 1, 2 primes (and no more) among the two numbers $(E - x_r)$, $(E - x_r - 2)$.............................(9*c*).

Thus D_r has only four possible values, viz.

$D_r = 0, 1, 2, 3$, (all positive)..............(9).

The preceding analysis shows that the several finite terms of the D_r series correspond exactly with the *positions* of the partitions (p, q), and indeed may be said to give the actual partitions (p, q) themselves; thus (noting that x_r is always a *prime*)—

When $y_r = x_r + 2$; $D_r = 1$ gives *one* partition $(x_r, E - x_r)$................(10*a*),

When $y_r = x_r + 4$; $D_r = 1$ gives *one* partition $(x_r, E - x_r)$, *or* $(x_r + 2, E - x_r - 2)$,
$D_r = 2$ gives *two* partitions $(x_r, E - x_r)$, *and* $(x_r + 2, E - x_r - 2)$...(10*b*),

When $r = 1$; $D_r = 1, 2, 3$ gives 1, 2, 3 partitions in which $p = 1, 3, 5, 7$ according as $E-1$, $E-3$, $E-5$, $E-7$ are *prime* $(= q)$, or *composite*...(10*c*),

In all cases; $D_r = 0$ shows the *absence* of a partition ....10*d*).

It will be seen that Haussner's formula for ν is—in its reduced form (6)—equivalent to an *enumeration* of the actual partitions. Each partition is thus counted *twice* when $p \neq q$—viz., once as $(p + q)$, and *again* as $(q + p)$—so that the number ν is only the *half* of the above enumeration, *i.e.* $\nu = \frac{1}{2}\Sigma_r D_r$ when $p \neq q$: but, when $\frac{1}{2}E = n = prime$, there is also one *unique* partition $E = n + n$. This accounts for the appearance of the factor $\frac{1}{2}$ in the formula (2), (6), and also for the introduction of j to meet the case of $\frac{1}{2}E = prime$. This shows further [along with Result (9)] that

Haussner's formula (2) always gives $\nu =$ a *positive integer, or zero*...(11).

Even, in its reduced form, Haussner's formula is of little practical use for computing the value of ν *a priori*, as it involves practically forming the partitions themselves, and then counting them, one by one, as formed.

8. *Can a ν-formula prove Goldbach's Theorem?*

It would seem at first that any exact expression for the number of partitions (ν) should contain in itself (implicitly) a proof of Goldbach's Theorem (if true). For this purpose it would be necessary and sufficient to show that the formula would always give a numerical result (ν) as follows:

1°. Rational; 2°. A + number; 3°. An integer; 4°. Finite (*i.e.* >0)...(12).

It might of course happen that the formula was too intractable* to yield proof (or disproof) of these properties readily.

8*a*. *Sylvester's formula.* It seems probable that Sylvester's formula must have been of the kind that was intractable (at the time) in at least one of the above points.

8*b*. *Haussner's formula.* It has been shown above [Result (11)] that Haussner's formula satisfies the *first three* of the above conditions. For the fourth condition it would suffice to show that *at least one* of the D_r terms of the reduced formula (6) was necessarily *finite* (*i.e.* > 0) for all values of E: but, as it has been shown above (Art. 7) that this appears to involve finding the actual partition (p, q) corresponding to that D_r, this *amounts to restating Goldbach's Theorem* itself in other words: so that the expectation of deducing the theorem from the reduced form (6) of this ν-formula seems illusory.

It may be, however, that this ν-formula, treated in some other way, may be shown to yield ν always finite; *e.g.* taking the value of D_r in Result (5) it may be† possible to show that

$$\Sigma P(E-x_r) > \Sigma P(E-y_r),\ [r=1 \text{ to } r=2n-1]\ldots(13).$$

9. *Variation of ν.* If Haussner's Table II. of the values of ν be examined, the variation of ν, as E increases, will appear at first sight to be quite irregular: but, if the numbers E be arranged in certain classes, a certain amount of regularity becomes manifest. The two following general rules have been propounded by‡ M. E. Ripert; they are based on Haussner's Table II. (extending to $E=5000$) and on a number of additional specially selected values of E, for which

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* Two Papers in *Nachrichten d. K. Gesell. d. Wissensch. zu Göttingen*, one by Herr P. Stäckel (1896), one by Herr E. Landau (1900), contain attempts at approximate formulæ for ν. They are difficult Papers, and do not appear to yield conclusive results.

† The author has not succeeded in doing this.

‡ See his Papers quoted in Art. 2.

the value of ν has been worked out by himself (extending to $E = 50816$).

i. The prime factors of E have a marked effect on ν: when these factors are few and large, ν is relatively small; when they are numerous and small, ν is relatively large..(14*a*).

ii. The powers of 2, when contained in E, have the general effect of depressing ν........ ....(14*b*).

These rules are unfortunately wanting in definiteness: the more definite rules, as yet proposed, appear however to be *not really general*; among these the following are of considerable generality.

1°. If $E = 2^k m$, then—

ν increases, as k increases, fairly steadily (m constant).............(15).

2°. If $E = 2^k.mp$ (p an odd prime)

ν increases, as p increases ($2^k.m$ constant), but with fluctuations...(16).

3°. ν is a local maximum, usually, when $E = 6m$ (> 30).............(17).

4°. ν is a local minimum, usually, when $E = 2p, 4p, 8p$ (p a prime) ..(18).

5°. ν is a local grand maximum, usually, when $E = 2\Pi(\Omega)$ ....(19), where $\Pi(\Omega) = 1.3.5.7.9....\Omega$ = product of successive odd numbers.

6°. ν is a local grand maximum, usually, when $E = 2\varpi(p)$..........(20), where $\varpi(p) = 1.3.5.7.11....p$, = product of successive odd primes [$p > 5$]; or (more precisely) $\nu_E > \nu_{E-2x}$, for all values of $x \not> \frac{1}{2}E$...............(20*a*), where ν_X denotes the number of solutions of $X = p + q$.

7°. ν is a local grand minimum, usually, when $E = 2^k$ (k a prime > 5)..(21).

Of these rules, Nos. 1°, 2 are due to the present writer; No. 3° is due to Herr Cantor; Nos. 4, 5, to M. Aubry; Nos. 6°, 7° to M. Ripert; but—though some of them are known to be satisfied up to $E = 5000$, and in some cases to much higher limits—they are probably liable to exceptions (under the operation of the more general rules Nos. i., ii. above) among high numbers; indeed M. Ripert has quoted exceptions to Rule 3°.

$E =$	10008; 8.9.139;	10010 2.5.7.11.13	$E =$	170166; 2.3.79.359;	170170; 2.5.7.11.13.17;	170172; 4.9.29.163;
$\nu =$	193;	195	$\nu =$	1867;	1892;	1855;

wherein $\nu_{6m} <$ one of $\nu_{6m+2}, \nu_{6m+4}, \nu_{6m-2}$, evidently in consequence of the predominance of small primes in the even numbers $E = 6m + 2$, $6m + 4$, $6m - 2$ (under the operation of Rule i).

[Other rules of less generality, proposed by some of the writers mentioned, are not quoted here, as not being of sufficient generality].

10. *Upper limit of ν.* The number n, defined by

n=number of primes from $\frac{1}{2}E$ to E (including $\frac{1}{2}E$ when prime),

is evidently an absolute upper limit to ν, and the question arises whether any, and (if so) what kind of, even numbers (E) can give $\nu=n$. An examination of Haussner's Tab. II. shows that the following are the only cases (up to $E \not> 5000$).

E =	2,	4,	6,	8,	10,	12,	14,	16,	18;	24,	30,	36,	42,	48,	60,	66,	90,	210;
$\nu=n$ =	1	2	2	2	2	2	3	2	3;	4	4	4	5	6	7	7	10	19;

though it is nearly approached by many other numbers, so long as E is small; this is apparently due to the greater frequency of primes among the small numbers. In fact the Table shows clearly that the excess of n over ν increases on the whole as E increases and becomes large when E is large. Thus accepting Messrs Aubry's and Ripert's results (Nos. 4°, 5°, above) that ν is a local grand maximum when $E=2\pi(\Omega)$ or $2\varpi(p)$, the falling off from the maximum possible (*i.e.* n) is seen to become pretty large even among these maxima as E increases,

E =	2.3,	2.3.5,	2.3.5.7,	2.3.5.7.9,	2.3.5.7.11,	2.3.5.7.11.13;
n (Limit) =	2	4	19	123	152	1493
ν (Actual) =	2	4	19	92	115	901

11. *Local Grand Minima.* If these minima, which occur (under M. Ripert's rule 7°) when $E=2^k$ (k prime), be examined, it will be seen that ν increases steadily as E increases (at any rate up to $E=2^{13}$) thus—

E =	2,	4,	8,	32,	128,	2048,	8192;
ν =	1,	2,	2,	3,	4,	25,	96;

and Haussner's Tab. II. shows that

The ν of $E=(2^k+x)$ is never < the ν of $E=2^k$ (up to 5000).........(22).

Moreover, it has been shown (Art. 4*b*, and Tab. III.) that Goldbach's Theorem certainly holds for all the numbers $E=2^n$ up to $n=28$ (*i.e.* over 268 million). All these results together seem to be pretty strong (numerical) evidence of the general truth of Golbach's theorem.

12. *Prime-twins theorem.* Two primes differing by 2 may be styled *Prime-twins*, and the smaller and larger prime of such a pair may be styled for shortness a *Minor Twin* and *Major Twin* respectively. Then, from Haussner's Tab. I., it

appears that the following curious theorem about even numbers of form $E=6m$ is probably true; (it is certainly true up to $E \not> 3000$).

Every multiple of 6 ($E=6m$) has at least one (and usually several) minor twins, and also at least one (and usually several) major twins among the primes p, q of its Goldbach partitions ....(22).

In fact, multiples of 6 ($E=6m$) have in most cases at least one *pair* (and usually several pairs) of prime-twins (p, q) among their Goldbach partitions: there are only 9 exceptions to this property (up to $E \not> 3000$), viz.

$$E = 6m = 96,\ 402,\ 516,\ 786,\ 906,\ 1116,\ 1146,\ 1266,\ 1356.$$

Now the whole series of even integers may be written in groups of three in form

$$E = 6m - 2,\ 6m,\ 6m + 2,$$

and it will be seen that the above theorem (22), if true for all numbers of form $E=6m$, involves the truth of Goldbach's theorem for all the remaining even numbers, these being all included in the forms $E=6m \mp 2$.

13. *Connexion with "sum-factors" problems.* There is an intimate connexion between Goldbach's Theorem and one of the "Sum-factors" Problems.

Let $\int N$ denote the sum of *all* the *divisors* of N (including 1 and N).

Let σN denote the sum of *all* the *sub-factors* of N (including 1, but not N).

Then $$\sigma N = \int N - N.$$

Now, let $N=pq$ (p, q being primes > 1, and $p \neq q$). Then

$$\int N = 1 + p + q + pq, \quad \sigma N = 1 + p + q.$$

Now, let it be proposed to find a number N, which shall have its sum-factors $\sigma N = X$, a (given) *odd* number. Goldbach's Theorem gives an easy solution of this (when $X > 7$).

Assume $N=pq$ (p, q being primes > 2, and $p \neq q$); to find p, q.

Then, by the above, $p + q + 1 = \sigma N = X$,

$$\therefore p + q = X - 1, \text{ an } \textit{even} \text{ number.........(23).}$$

Hence, every partition (p, q) of the *even* number ($X-1$) into a sum of two odd primes (p, q) such that $p \neq q$ and p, q each > 1, gives a value of $N=pq$, such that $\sigma N = X$, as

required. Hence the number of solutions (of form* $N=pq$) is the same as the number ν of Goldbach partitions of $(N-1)$, [after excluding the cases of $p=q$, and p or $q=1$, if they occur among the Goldbach partitions].

Thus the two problems are intimately connected. The possibility of either involves that of the other (excluding, of course, the two cases excepted above, which do not occur in the "sum-factors" problem).

14. *Analogues of Goldbach's Theorem.* Several theorems as to odd and even numbers, analogous to Goldbach's theorem, have been propounded as probably generally true; but are, as yet, without strict proof (the evidence being in fact only numerical verification). Such are the following: references are given to the author's name and original publication (so far as known to the present author).

i. *As to even numbers* (E). [Here p, q denote primes].

1°. *A. de Polignac* (Ed. Lucas's *Théorie des Nombres*, Vol. I, Paris, 1891, Art. 198).

Every even number is the difference of two primes, or $E=q-p$...(24).

2°. *A. de Rocquigny* (*L'Interméd. des Mathém.*, t. v., 1898, p. 268, Q. 1408, and t. vi., 1899, p. 164).

Every multiple of 6 is the difference of two primes of form $(6\varpi+1)$...(25).

ii. *As to odd numbers* (Ω). [Here p, q, q' denote primes].

1°. *de Lagrange*, (*Oeuvres*, t. iii., 1869, last page)

$p=q+2q'$; where $p=4\varpi-1$, $q=4\kappa+1$, $q'=4\kappa'+1$.........(26).

2°. *E. Lemoine*, (*L'Interméd. des Mathém.*, t. i, 1894, p. 179, Q. 30]

Every odd number $\Omega=p+2q$......................(27),

3°. [Reference wanting].

Every odd number $\Omega=p-2q$..........................(28),

Here follow some theorems of a more complex kind—

1°, 2°, 3°, *L. Ripert*, (*L'Interméd des Mathém.* t. x., 1903, p. 68, Q. 2541)

1°. $E=\omega^x+p$ (up to $E \not> 10000$)..................................(29*a*).

2°. $\Omega=\epsilon^x+p$ (up to $E \not> 10000$, except $\Omega=3, 127, 1549$)...(29*b*).

3°. $N=a^x+p$ (up to $N \not> 10000$, except $N=1549$)............(29*c*).

[Herein ω, Ω denote odd numbers; ϵ, E denote even numbers; N, a any integers; $x>1$; p any prime.]

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* There may of course be solutions with *other forms* of N; but the form $N=pq$ is alone considered here. It will be seen that the results N are all odd numbers (since p, q are both odd).

Class	E	n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Hyper-Even.	2^n	$p=$	1	1,2	1,3	3	1,3	3	1,19	5	3	3	19	3	1,13	3	19	17	1,13	5	1,19	3	19	3	37	3	61	5	
		$m=$	0	1	1	3	3	9	3	21	15	9	15	21	3	45	135	75	15	99	93	99	315	105	105	15	75	339	117
	3.2^n	$p=$	1,3	1,5	1,5	1,5	7	1,11	1,5	7	5	5	1,11	7	5	13	5	11	7	1,13	11	7	7						
		$m=$	0	1	1	5	5	7	1	5	29	13	11	53	35	95	49	83	109	25	1	23	95	155					
	5.2^n	$p=$	3	1,3	3	1,7	3	3	23	1,3	3	1,7	17	1,3	11	1,19	29	7	3	1,109	53	3							
		$m=$	0	3	3	3	9	3	27	21	3	57	33	63	3	33	51	69	99	93	39	177	3						
	7.2^n	$p=$	1,3	5	3	3	1,13	5	13	3	1,3	17	13	3	13	17	3	5	1,43	5	3	23							
		$m=$	0	3	9	3	15	27	9	15	9	93	9	33	15	123	111	33	219	45	291	135	135						
	9.2^n	$p=$	1,5	5	1,5	5	5	5	1,23	7	5	7	5	7	1,7	5	1,19	13	7	29	13								
		$m=$	2	1	5	1	5	5	55	1	7	41	65	19	7	29	47	125	217	149	73	29							
	11.2^n	$p=$	3	1,3	5	3	3	3	41	13	41	3	17	3	5	3	41	19	41	7	59								
		$m=$	0	9	3	9	3	15	57	81	27	9	87	45	177	105	63	9	117	189	267	39							
Barely Even.	$2(2^n-11)$	$p=$	.	.	.	3	1,5	3	1,5	3	5	23	1,17	3	1,13	29	17	41	1,11	5	5	47	5	5	11	17	5		
		$m=$	.	.	.	0	2	0	10	6	2	0	26	6	10	54	40	132	2	0	74	6	118	126	26	216	80	324	
	$2(2^n+11)$	$p=$	3	1,7	1,7	1,7	3	1,11	1,7	11	7	1,7	7	5	37	1,7	7	23	13	1,23	151	5	7	7	97	61	7		
		$m=$	0	2	0	4	0	4	0	4	0	4	30	50	114	26	0	4	60	76	42	14	66	14	0	106	60	38	
	$2(2^n-9)$	$p=$	.	.	.	1,3	3	1,3	5	3	23	1,3	5	3	3	1,31	71	13	5	1,13	41	1,3	113	3	59	31	83		
		$m=$	.	.	.	0	0	12	12	14	0	6	0	66	36	6	42	30	0	378	90	60	0	24	18	234	156	78	
	$2(2^n+9)$	$p=$	3	3	3	3	3	7	3	7	3	3	3	31	41	3	1,7	19	11	19	5	1,37	3	3	107	67	3		
		$m=$	0	0	0	6	0	0	0	42	0	0	30	6	30	24	6	6	222	0	54	198	288	96	0	306	348	294	
	$2(2^n-7)$	$p=$	.	.	1	1,5	3	1,5	1,3	7	1,13	5	3	7	7	5	1,3	17	3	5	3	5	3	1,7	3	47	13		
		$m=$	.	.	0	2	6	4	18	8	18	4	12	10	6	44	42	8	6	10	60	20	114	106	162	82	42	100	
	$2(2^n+7)$	$p=$	1,5	3	1,7	3	5	3	1,7	3	5	23	11	39	17	3	7	23	5	41	1,7	23	17	3	17	53	19		
		$m=$	2	0	2	0	2	0	4	0	28	0	26	24	20	30	4	0	70	0	52	0	128	288	22	354	200	162	
	$2(2^n-5)$	$p=$	.	.	1,3	3	1,7	5	5	3	1,5	11	7	3	5	41	5	3	1,7	17	7	11	7	5	7	29	17		
		$m=$	.	.	0	0	4	0	14	0	16	0	26	0	76	150	70	12	4	0	26	0	64	930	16	618	154	0	
	$2(2^n+5)$	$p=$	1,3	1,5	3	1,5	1,3	1,7	3	1,13	1,3	5	7	11	13	7	3	11	1,3	11	3	19	13	1,31	13	59	37		
		$m=$	0	2	0	2	0	2	6	10	30	10	0	10	36	28	24	2	36	2	96	32	66	62	336	38	66	10	
	$2(2^n-3)$	$p=$	.	1	3	3	5	13	11	3	5	3	11	7	17	13	11	3	5	13	11	3	11	31	11	43	101		
		$m=$	.	0	0	0	0	0	12	24	0	0	18	0	42	0	78	54	60	90	24	0	108	0	12	0	270	408	
	$2(2^n+3)$	$p=$	3	1,3	3	1,7	3	3	5	19	11	1,37	3	7	29	3	3	7	3	7	11	61	23	43	93	67	11		
		$m=$	0	0	0	0	6	0	0	18	6	6	12	0	48	24	0	0	246	0	60	30	132	90	72	288	414	90	
	$2(2^n-1)$	$p=$	1	1,3	1.3	1,7	1,3	13	3	1,7	1,3	7	3	11	1,13	17	13	7	3	17	1,3	7	1,31	13	1,31	37	3		
		$m=$	0	0	0	2	0	4	0	14	12	10	36	4	0	34	120	16	0	4	0	26	18	16	156	74	210	170	
	$2(2^n+1)$	$p=$	1,3	3	1,5	3	5	3	1,7	5	5	11	5	3	5	53	1,17	3	7	3	5	11	5	17	5	41	7		
		$m=$	0	0	2	0	4	6	2	0	10	6	20	174	26	36	20	0	230	6	20	6	70	114	28	360	40	336	

Tab. II.

Class	E	n =	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Semi-Hyper-Even.	$(4.3)^n$	$p =$	1,5	5	5	5	5	5									
		$m =$	1	1	43	31	73	289									
	$(4.5)^n$	$p =$	1,3	3	7	23	3										
		$m =$	3	27	57	21	69										
	2.10^n	$p =$	1,3	1,3	1,3	3	1,67	7	1,19								
		$m =$	3	3	9	69	129	39	261	213							
	$(2.3)^n$	$p =$	1,3	5	5	5	17	7	17	7							
		$m =$	0	1	1	5	35	119	1	155	55						
	$(2.5)^n$	$p =$	3	3	3	59	11	17	29	11							
		$m =$	0	3	9	81	123	57	87	243							
	$(2.7)^n$	$p =$	1,3	3	3	23	13	17									
		$m =$	0	9	51	51	129	225									
	$(2.9)^n$	$p =$	1,5	7	5	5	7										
		$m =$	2	5	37	55	73										
	$(2.11)^n$	$p =$	3	5	17	17	83										
		$m =$	0	9	27	75	15										
	$2^n(2^n-1)$	$p =$	1	1,5	3	1,7	1,9	5	3	11	1,13	1,13	37				
		$m =$	0	1	9	7	75	13	39	7	165	17	81	11			
	$2^n(2^n+1)$	$p =$	1,3	1,3	1,5	1,3	5	1,3	19	3	5	1,31	5				
		$m =$	0	3	5	27	19	51	13	117	35	57	7	213			
Barely Even.	2.3^n	$p =$	1,3	1,5	1,7	5	7	5	1,11	1,13	7	5	23	1,5	5	8	16
		$m =$	0	2	4	2	14	10	26	8	80	20	20	110	64		
	2.5^n	$p =$	3	3	11	1,13	3	1,3	23	3	3						
		$m =$	0	6	12	6	42	18	66	126	144						
	2.7^n	$p =$	1,3	1,19	3	1,3	1,13	67	3								
		$m =$	0	12	6	120	114	30	186	102							
	2.11^n	$p =$	3	1,3	3	13	5	73									
		$m =$	0	18	30	12	42	72									

Tab. III.

Class	E	p
Hyper-Even.	2^{28}	3063803
	3.2^{23}	97303
	3.2^{24}	330791
	5.2^{22}	186229
	5.2^{23}	1174259
	5.2^{24}	1376503
	7.2^{22}	384737
	7.2^{23}	5567503
	9.2^{21}	2297
	9.2^{22}	362449
	9.2^{23}	199999
	11.2^{21}	49031
	11.2^{22}	84067
	11.2^{23}	4525751
Semi-Hyper-Even.	$(4.3)^7$	101
	$(4.5)^6$	2641739
	$(2.3)^{10}$	4693727
	$(2.9)^6$	457057
	$2^{13}(2^{13}-1)$	1102903
	$2^{13}(2^{13}+1)$	7129
Barely Even.	2.3^{16}	1326761
	2.5^{10}	136033
	2.5^{11}	1303139
	2.7^9	2641153
	2.11^7	1202441

ON AN EXPANSION OF AN ARBITRARY FUNCTION IN A SERIES OF BESSEL FUNCTIONS.

By *H. Bateman.*

In a recent investigation* on the expansion

$$\psi(x) = c_1 J_1(x) + c_3 J_3(x) + c_5 J_5(x) + \ldots,$$

where
$$c_n = 2n \int_0^\infty \frac{\phi(z)}{z} J_n(z)\, dz, \quad (n \text{ odd}),$$

W. Kapteyn shows that the above equation will be exact if the function $\phi(x)$ satisfies the equation

$$\int_0^\infty \frac{J_1(t)}{t} [\phi(t+\gamma) + \phi(t-\gamma)]\, dt = 2\frac{d}{d\gamma}\phi(\gamma),$$

and proposes the problem of finding all the functions ϕ which satisfy this equation.

With the view of obtaining a solution of this problem I have studied the properties of a particular class of series of the above form, and have obtained a result which seems to have an important bearing on the question.

We shall consider the case in which the series

$$\Sigma \sqrt{(x)}\, c_{2n+1} J_{2n+1}(x) = \sqrt{(x)}\, \phi(x)$$

is absolutely and uniformly convergent† for all real values of x, and shall show that a function which can be defined by means of such a series can be represented in the form

$$\phi(x) = x \int_0^1 J_0(xt) f(t)\, dt.$$

For large positive values of x the function $J_n(x)$ approximates to the value‡

$$\left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos\{x - (n + \tfrac{1}{2}) \tfrac{1}{2}\pi\},$$

accordingly if the above series is to be absolutely and uniformly

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* *Messenger*, Vol. XXXV., No. 8, p. 122. The expansion was originally given by Webb, Vol. XXXIII., p. 55.

† When this is the case we can certainly determine the coefficients c_{2n+1} by multiplying by $x^{-\frac{1}{2}} J_{2n+1}(x)$ and integrating between $-\infty$ and $+\infty$, for the series satisfies the rule laid down by G. H. Hardy (*Messenger*, Vol XXXV., p. 126).

‡ See Whittaker's *Analysis*, p. 290.

convergent for large values of x, the series

$$\Sigma \,|\, c_{2n+1} \,|$$

must be convergent.

Now consider the function

$$\psi(t) = \sum_{n=0}^{\infty} c_{2n+1} P_n(1-2t^2),$$

where $P_n(\mu)$ is the Legendre polynomial. Since the value of $P_n(\cos\theta)$ lies between -1 and $+1$, the above series will be absolutely and uniformly convergent for all values of t between 0 and 1, accordingly we may calculate the integral

$$\int_0^1 J_0(xt)\, t\psi(t)\, dt$$

by multiplying the series by $tJ_0(xt)$ and integrating term by term.

In a former paper* I showed that

$$(1) \qquad \sum_0^{\infty} 2(2n+1) J_{2n+1}(x) P_n(1-2t^2) = xJ_0(xt),$$

and this series is uniformly convergent for values of t lying between 0 and 1, hence by the usual rule for obtaining the coefficients in an expansion in terms of the Legendre polynomials, we have

$$\int_0^1 xJ_0(xt)\, tP_n(1-2t^2)\, dt = J_{2n+1}(x).$$

Hence

$$x\int_0^1 J_0(xt)\, t\psi(t)\, dt = \sum_0^{\infty} c_{2n+1} \int_0^1 xJ_0(xt)\, tP_n(1-2t^2)\, dt$$

$$= \sum_0^{\infty} c_{2n+1} J_{2n+1}(x) = \phi(x).$$

Thus the function $\phi(x)$ can be represented by means of a definite integral of the form

$$\phi(x) = x\int_0^1 J_0(xt)\, t\psi(t)\, dt.$$

The function $\psi(t)$ may be obtained directly from the function $\phi(x)$ by means of a formula due to Hankel†

* *Messenger*, Vol. XXXIII., pp. 33–40. The result is obtained by putting $m=0$, $\sin\theta = t$, $\omega = \frac{1}{2}\pi$ in equation (2). It should be noticed that equation (3) is given incorrectly, the numbers $2m+1$, $2m+3$ should be in the numerator.

† *Math. Ann.*, Bd. 8, p. 482, 1875.

If $$\Psi(x) = \int_0^\infty J_0(xt)\, t\chi(t)\, dt,$$

then $$\chi(t) = \int_0^\infty J_0(xt)\, x\Psi(x)\, dx.$$

Writing $$\chi(t) = \psi(t), \quad \text{if } t < 1,$$
$$\chi(t) = 0, \quad \text{if } t > 1,$$

we shall have $\Psi(x) = \dfrac{\phi(x)}{x}$, and the formula gives

$$\int_0^\infty \phi(x)\, J_0(xt) = \psi(t), \quad \text{if } t < 1,$$
$$= 0, \quad \text{if } t > 1.$$

We have thus found a *necessary* condition to be satisfied by a function $\phi(x)$ in order that its expansion in series of Bessel functions may converge in the manner stated above.

Let us examine the case of the function $\sin(xz)$, which is quoted by Kapteyn as a function for which Webb's expansion is not possible when $|z| > 1$.

Schafheitlin* has shown that

$$\int_0^\infty J_0(xt) \sin(xz)\, dx = \frac{1}{\sqrt{(z^2 - t^2)}}, \quad \text{if } t^2 < z^2,$$
$$= 0, \quad \text{if } t^2 > z^2,$$

accordingly if $z^2 < 1$ the condition is satisfied, but if $z^2 > 1$ it is not. Moreover, the series which is obtained by calculating the coefficients according to the rule

$$c_n = 2n \int_0^\infty \frac{\phi(z)}{z} J_n(z)\, dz,$$

viz. $$2\left[\tan\tfrac{1}{2}\theta\, J_1(x) - \tan^3\tfrac{1}{2}\theta\, J_3(x) + \tan^5\tfrac{1}{2}\theta\, J_5(x) - \ldots\right],$$

where $$z = \operatorname{cosec}\theta,$$

does satisfy the conditions of convergence which we have laid down, it therefore represents a function of the form

$$x\int_0^1 J_0(xt)\, t\psi(t)\, dt,$$

* *Math. Ann.*, Bd. XXX.

but it does not represent $\sin(x \operatorname{cosec}\theta)$, because this function cannot be so expressed.

§ 2. We may obtain a simple rule for determining whether the equation

$$\phi(x) = c_1 J_1(x) + c_3 J_3(x) + c_5 J_5(x) + \ldots,$$

where
$$c_n = 2n \int_0^\infty \frac{\phi(z)}{z} J_n(z)\, dz \qquad (n \text{ odd})$$

is correct by comparing it with Neumann's expansion which is known to represent $\phi(x)$ within a certain circle.

The coefficient c_n is determined by the formula

$$c_n = 2\left\{n\phi'(0) + \frac{n(n^2-1^2)}{3!}\phi'''(0) + \ldots 2^{n-1}\phi^{(n)}(0)\right\} \qquad (n \text{ odd}),$$

but this is exactly the value which we should obtain if we calculate the above integral by expanding $\phi(z)$ in ascending powers of x and integrating term-by-term as if the divergent integral

$$\int_0^\infty J_n(ax)\, x^q dx$$

had the value*

$$\frac{2^q}{a^{q+1}} \frac{\Pi\{\frac{1}{2}(n+q-1)\}}{\Pi\{\frac{1}{2}(n-q-1)\}}.$$

In practice it is often easy to recognize at once whether the correct value of the integral is obtained by evaluating it in this way, especially if the function to be expanded contains a parameter.

For example, if $\phi(x) = J_m(xz)$, $z^2 < 1$, m odd, we have†

$$c_n = 2n \int_0^\infty J_m(xz)\, J_n(x) \frac{dx}{x}$$

$$= nz^m \frac{\Pi[\{\frac{1}{2}(m+n)\}-1]}{\Pi[\{\frac{1}{2}(n-m)\}\Pi(m)]} F\{\tfrac{1}{2}(m+n),\ \tfrac{1}{2}(m-n),\ m+1,\ z^2\}\ (n>m),$$

and this is evidently the value obtained by the above rule. Hence, if $z^2 < 1$,

$$\frac{J_m(xz)}{z^m} = \sum_{r=0}^\infty (m+2r) \frac{\Pi(m+r-1)}{\Pi(r)\,\Pi(m)} F(m+r,\ -r,\ m+1,\ z^2) J_{m+2r}(x).$$

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* A theory of divergent integrals is suggested by G. H. Hardy, *Quarterly Journal*, Vol. xxxv, p. 22.

† Schafheitlin, *Math. Ann.*, Bd. xxx.

It should be noticed that the quantity

$$\frac{2^q}{a^{q+1}} \frac{\Pi\{\frac{1}{2}(n+q-1)\}}{\Pi\{\frac{1}{2}(n-q-1)\}}$$

is zero if $q-n$ is an odd positive integer. This explains why an integral of the type

$$\int_0^\infty J_n(ax) f(bx)\, dx$$

is often a polynomial in b/a. When $f(bx) \equiv J_0(bx)$ and n is odd, we get the Legendre polynomial $(b^2 < a^2)$, and I think I am right in saying that the integral

$$\int_\iota J_a(\alpha x) J_b(\beta x) J_c(\gamma x) \ldots J_l(\lambda x) J_{a+b+\cdots l+2n+1}(x)\, dx$$

is a polynomial of degree $(2n+a+b+\ldots l)$ in α, β, ..., λ, provided the quantities $\alpha \ldots \lambda$ are subject to the inequalities

$$\pm\alpha \pm \beta \ldots \pm \lambda < 1.$$

§ 3. We may obtain another expression for the sum of the series $\sum\limits_0^\infty nJ_n(x) J_n(z)$ by substituting for $J_n(z)$ the definite integral

$$J_{2m+1}(z) = \int_0^1 zJ_0(zt)\, P_m(1-2t^2)\, tdt$$

obtained above; the result is

$$\sum_0^\infty (2m+1) J_{2m+1}(x) \int_0^1 zJ_0(zt)\, P_m(1-2t^2)\, tdt$$
$$= 2xz \int_0^1 J_0(zt)\, J_0(xt)\, tdt$$

on account of relation (1).

Hence, putting $t^2 = \xi$, we have

$$\Sigma(2m+1) J_{2m+1}(x) J_{2m+1}(z) = xz \int_0^1 J_0(x\sqrt{\xi})\, J_0(z\sqrt{\xi})\, d\xi.$$

§ 4. Another expansion of some interest is that in which the Bessel functions are of order $n+\frac{1}{2}$, and if

$$f(x) = \sum_0^\infty a_n \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}},$$

the corresponding rule for determining the coefficients is

$$a_n = (n+\tfrac{1}{2})\int_{-\infty}^{\infty} f(x)\frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}\,dx,$$

but here again it is not certain whether the series will converge to the sum $f(x)$.

Consider as before the particular case in which the series

$$\Sigma a_n \sqrt{(x)}\, J_{n+\frac{1}{2}}(x)$$

converges absolutely and uniformly* for all real values of x. Substituting the approximate value for $J_{n+\frac{1}{2}}(x)$ we see that the series

$$\Sigma\,|\,a_{2m}\,| \text{ and } \Sigma\,|\,a_{2m+1}\,|$$

must converge.

We now construct the function

$$\Sigma i^n a_n P_n(\mu) = \psi(\mu).$$

Since the series converges absolutely and uniformly, we have by the usual rule

$$i^n a_n = \tfrac{1}{2}(2n+1)\int_{-1}^{+1}\psi(\mu)\,P_n(\mu)\,d\mu.$$

Therefore

$$f(x) = \sum_0^\infty \{\tfrac{1}{2}(2n+1)\}(-i)^n \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}\int_{-1}^{+1}\psi(\mu)\,P_n(\mu)\,d\mu,$$

but it is well known† that

$$e^{-ix\mu} = \sum_0^\infty \sqrt{(2\pi)}\,(2n+1)\,(-i)^n \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}}\,P_n(\mu);$$

hence, on integrating term-by-term, we have

$$f(x) = \tfrac{1}{2}\sqrt{\left(\frac{1}{2\pi}\right)}\int_{-1}^{+1} e^{-ix\mu}\psi(\mu)\,d\mu,$$

and in order that the function $f(x)$ may be expanded in a series which converges in the specified way, it must be capable of being expressed by a definite integral of this type.

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* The above rule for determining the coefficients will then certainly apply.

† See Heine's *Handbuch der Kugelfunctionen*, or Rayleigh's *Sound*, Vol. II., p. 272.

The above equation may be solved for $\psi(\mu)$ by means of Fourier's theorem, and we have

$$\int_{-\infty}^{\infty} e^{-ix\mu} f(x)\,dx = \sqrt{(\tfrac{1}{2}\pi)}\,\psi(\mu) \quad \text{if } \mu^2 < 1,$$

$$0 \qquad \mu^2 > 1;$$

hence a necessary condition is that the function should satisfy the condition

$$\int_{-\infty}^{\infty} e^{-ix\mu} f(x)\,dx = 0, \qquad \mu^2 > 1.$$

If $f(x) = \dfrac{\sin xz}{x}$, the integral is zero for $\mu^2 > z^2$; hence, in order that the expansion may be of the required type, we must have $z^2 < 1$.

ON SOME PROPERTIES OF THE FUNCTION $\left(\omega^m, \dfrac{1}{1+r}\right)$.

By *H. Holden*, Shrewsbury School.

LET

$$\left(\omega^m, \frac{1}{1+r}\right) = \frac{1}{1+r} + \frac{\omega^m}{1+rg} + \frac{\omega^{2m}}{1+rg^2} + \ldots + \frac{\omega^{(p-2)m}}{1+rg^{p-2}},$$

where ω is a primitive root of $x^{p-1} = 1$,

g ,, ,, ,, $x^{p-1} \equiv 1 \pmod{p}$,

r is a root of $x^p = 1$,

p is a prime.

I. From the results in a previous paper (*Messenger*, Vol. XXXV., p. 73), it may be seen that, if $p = 4n+3$,

$$\left(\omega^{\frac{1}{2}(p-1)}, \frac{1}{1+r}\right) = 2/p \,.\, (\gamma_0 - \gamma_1)\, h,$$

$$\left(\omega^0, \frac{1}{1+r}\right) = \tfrac{1}{2}(p-1).$$

In the present paper it is proved that

$$\left(\omega^m, \frac{1}{1+r}\right)\left(\omega^{-m}, \frac{1}{1+r}\right) = 0,$$

if m is even and not divisible by $p-1$,

$$=p\Sigma_{\mu=1}^{\mu=p-1}\omega^{m\,\mathrm{ind}\,\mu}.\lambda_\mu,\ \text{if } m \text{ is odd},$$

where λ_μ is the number of solutions, for a given value of μ, of the congruence $k\mu+l\equiv 0$ (mod. p): k and l being positive integers not greater than $\frac{1}{2}(p-1)$.*

From this it is deduced that, if p is of form $4n+3$,

$$h^2=-\Sigma_{\mu=1}^{\mu=p-1}\mu/p.\lambda_\mu,$$

and $$bh=\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)}\mu/p.\lambda_\mu,$$

where b is the number of non-residues less than $\frac{1}{2}p$.

A more general formula is also given.

It is also proved that, if m is odd,

$$\left(\omega^m,\ \frac{1}{1+r}\right)(\omega^{-m},\ r)=p\Sigma_{\mu=1}^{\mu=p-1}\omega^{m\,\mathrm{ind}\,\mu}.\theta_\mu,$$

where θ_μ is number of solutions, for a given value of μ, of the congruence $2k\mu+1\equiv 0$ (mod. p): k being a positive integer less than $\frac{1}{2}p$.

It is deduced, if $p=4n+3$, that

$$-2/p.h=-h+2\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)}\mu/p.\theta_\mu=+h+2\Sigma_{\mu=\frac{1}{2}(p+1)}^{\mu=p-1}\mu/p.\theta_\mu,$$

where θ_μ is the number of solutions of $2k+\mu\equiv 0$ (mod. p).

A more general formula is also given.

II. To find the value of $\left(\omega^m,\ \frac{1}{1+r}\right)\left(\omega^{-m},\ \frac{1}{1+r}\right)$.

Calling this product P, we have, as in Bachmann, *Zahlentheorie*, Part 3, p. 86,

$$P=\Sigma_{\mu=1}^{\mu=p-1}\Sigma_{\mu'=1}^{\mu'=p-1}\omega^{m\,(\mathrm{ind}\,\mu-\mathrm{ind}\,\mu')}\frac{1}{1+r^\mu}.\frac{1}{1+r^{\mu'}}$$

$$=\Sigma_{\mu=1}^{\mu=p-1}\Sigma_{\mu'=1}^{\mu'=p-1}\omega^{m\,\mathrm{ind}\,\mu}\frac{1}{1+r^{\mu\mu'}}.\frac{1}{1+r^{\mu'}}$$

$$=\Sigma_{\mu=1}^{\mu=p-1}\omega^{m\,\mathrm{ind}\,\mu}\left\{\frac{1}{1+r^\mu}.\frac{1}{1+r}+\frac{1}{1+r^{2\mu}}.\frac{1}{1+r^2}+\ldots+\frac{1}{1+r^{(p-1)\mu}}.\frac{1}{1+r^{p-1}}\right\}.$$

---

* $\lambda_\mu=\Sigma(-1)^n\left[\frac{np}{2\mu}\right]$, where the limits of n are 1 and $2\,[\frac{1}{2}\mu]$, the square brackets indicating that the integral part of the enclosed fraction is to be taken.

The quantity in the bracket may be expressed with integral coefficients by means of the relation

$$\frac{1}{1+r^n} = \frac{1-r^n}{1-r^{2n}} = 1 + r^{2n} + r^{4n} + \ldots + r^{(p-1)n},$$

and therefore, as it remains unchanged when r^g is substituted for r, it is an integer, $\phi(\mu)$ say.

Therefore $$P = \Sigma_{\mu=1}^{\mu=p-1} \omega^{m \operatorname{ind} \mu} . \phi(\mu).$$

Each term, $\frac{1}{1+r^{\mu s}} \times \frac{1}{1+r^s}$, in the bracket may be written as

$$(1 + r^{2\mu s} + r^{4\mu s} + \ldots + r^{(p-1)\mu s})(1 + r^{2s} + r^{4s} + \ldots + r^{(p-1)s}).$$

In this product the first term is 1, and so is any term for which

$$2k\mu s + 2ls \equiv 0 \pmod{p},$$

or $$k\mu + l \equiv 0 \pmod{p},$$

where k and l are positive integers not greater than $\frac{1}{2}(p-1)$.

Let λ_μ be the number of solutions of the congruence $k\mu + l \equiv 0 \pmod{p}$ for a given value of μ; k and l satisfying the conditions mentioned above. Then considering the whole bracket, which consists of $p-1$ fractional terms, we get, on expanding,

$$p - 1 + (p-1)\lambda_\mu \text{ terms equal to } + 1.$$

But there are $(p-1)\{\frac{1}{2}(p+1)\}^2$ terms altogether, and therefore the remaining terms,

$$(p-1)\{\tfrac{1}{2}(p+1)\}^2 - (p-1)(1+\lambda_\mu)$$

in number, may be collected into groups of $p-1$ terms, each group being equal to -1, and therefore there will be $\{\frac{1}{2}(p+1)\}^2 - (1+\lambda_\mu)$ negative units; therefore

$$\begin{aligned}\phi(\mu) &= (p-1)(1+\lambda_\mu) - \{\tfrac{1}{2}(p+1)\}^2 + 1 + \lambda_\mu \\ &= p\lambda_\mu - \{\tfrac{1}{2}(p-1)\}^2,\end{aligned}$$

therefore

$$\begin{aligned}P &= \Sigma_{\mu=1}^{\mu=p-1} \omega^{m \operatorname{ind} \mu} \left[p\lambda_\mu - \{\tfrac{1}{2}(p-1)\}^2\right] \\ &= \Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind} \mu} \left[p\lambda_\mu - \{\tfrac{1}{2}(p-1)\}^2\right] \\ &\qquad + \omega^{m \operatorname{ind}(p-\mu)} \left[p\lambda_{p-\mu} - \{\tfrac{1}{2}(p-1)\}^2\right].\end{aligned}$$

It is easily seen that

$$\lambda_\mu + \lambda_{p-\mu} = \tfrac{1}{2}(p-1),$$

and that $$\operatorname{ind}(p-\mu) = \operatorname{ind}\mu \pm \tfrac{1}{2}(p-1).$$

Therefore

$$P = \Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu} \left[p\lambda_\mu - \{\tfrac{1}{2}(p-1)^2\}\right.$$
$$\left. + \omega^{m\{\operatorname{ind}\mu \pm \frac{1}{2}(p-1)\}} \left[\tfrac{1}{2}\{p(p-1)\} - p\lambda_\mu - \{\tfrac{1}{2}(p-1)\}^2\right]\right]$$
$$= \Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu} \left\{p\lambda_\mu - [\tfrac{1}{2}(p-1)]^2\right.$$
$$\left. + (-1)^m \left[\tfrac{1}{2}p(p-1) - p\lambda_\mu - \{\tfrac{1}{2}(p-1)\}^2\right]\right\},$$

If m is even, this gives

$$P = \tfrac{1}{2}(p-1)\,\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu}$$
$$= \{\tfrac{1}{2}(p-1)\}^2 \text{ if } m \text{ is divisible by } p-1$$
$$= 0 \text{ if } m \text{ is not divisible by } p-1,$$

The first case is evident, as each term in the sum is $+1$; for the second case we have

$$\Sigma_{\mu=1}^{\mu=p-1} \omega^{m \operatorname{ind}\mu} = 0,$$

Therefore

$$\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu} + \Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}(p-\mu)} = 0,$$

therefore

$$\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu} + (-1)^m\,\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu} = 0,$$

therefore $$\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu} = 0,$$

when m is even and not divisible by $p-1$.

If m is odd the general equation reduces to

$$P = \Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{m \operatorname{ind}\mu} \left[2p\lambda_\mu - \tfrac{1}{2}\{p(p-1)\}\right].$$

So far the results hold for all prime values of p.

III. If p is a prime of form $4n+3$, then putting $m=\frac{1}{2}(p-1)$, an odd number, and remembering that

$$\left(\omega^{\frac{1}{2}(p-1)}, \frac{1}{1+r}\right)=\left(\omega^{-\frac{1}{2}(p-1)}, \frac{1}{1+r}\right)=2/p\,(\gamma_0-\gamma_1)\,h,$$

we have

$$-ph^2=\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)}(-1)^{\operatorname{ind}\mu}\left[2p\lambda_\mu-\tfrac{1}{2}\{p\,(p-1)\}\right],$$

$$h^2=\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)}\mu/p\left[\{\tfrac{1}{2}(p-1)\}-2\lambda_\mu\right]$$

$$=\tfrac{1}{2}(p-1)\,.\,h-2\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)}\mu/p\,.\,\lambda_\mu,$$

or

$$bh=\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)}\mu/p\,.\,\lambda_\mu,$$

if b is the number of non-residues less than $\frac{1}{2}p$.

Starting from the equation

$$P=\Sigma_{\mu=1}^{\mu=p-1}\omega^{m\operatorname{ind}\mu}\left[p\lambda_\mu-\{\tfrac{1}{2}(p-1)\}^2\right],$$

the same substitutions give

$$-ph^2=\Sigma_{\mu=1}^{\mu=p-1}\mu/p\left[p\lambda_\mu-\{\tfrac{1}{2}(p-1)\}^2\right],$$

or

$$h^2=-\,\Sigma_{\mu=1}^{\mu=p-1}\mu/p\,.\,\lambda_\mu.$$

IV. The product $\left(\omega^m, \frac{1}{1+r^t}\right)\left(\omega^{-m}, \frac{1}{1+r^t}\right)$ may be evaluated in the same way, and, as $\left(\omega^m, \frac{1}{1+r^s}\right)=\omega^{-m\operatorname{ind}s}\left(\omega^m, \frac{1}{1+r}\right)$, we should get the more general results

$$s/p\,.\,t/p\,.\,h^2=\tfrac{1}{2}(p-1)\,.\,h-2\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)}\mu/p\,.\,\lambda_\mu,$$

$$s/p\,.\,t/p\,.\,h^2=-\,\Sigma_{\mu=1}^{\mu=p-1}\mu/p\,.\,\lambda_\mu,$$

where λ_μ is the number of solutions, for a given value of μ, of the congruence $ks\mu+lt\equiv 0 \pmod{p}$; k and l being positive integers less than $\frac{1}{2}p$.

V. The value of $\left(\omega^m, \dfrac{1}{1+r}\right)(\omega^{-m}, r)$ may be found in the same way. Calling this product P, we have

$$P = \Sigma_{\mu=1}^{\mu=p-1} \Sigma_{\mu'=1}^{\mu'=p-1} \omega^{m \operatorname{ind} \mu} \cdot \frac{r^{\mu'}}{1 + r^{\mu\mu'}},$$

$$= \Sigma_{\mu=1}^{\mu=p-1} \omega^{m \operatorname{ind} \mu} \left\{ \frac{r}{1+r^{\mu}} + \frac{r^2}{1+r^{2\mu}} + \ldots + \frac{r^{p-1}}{1+r^{(p-1)\mu}} \right\};$$

as before, the quantity in the bracket is an integer $\phi(\mu)$, as

$$\begin{aligned}\phi(\mu) = r &\{1 + r^{2\mu} + \ldots + r^{(p-1)\mu}\} \\ &+ r^2 \{1 + r^{4\mu} + \ldots + r^{(2p-2)\mu}\} \\ &+ \\ &\vdots \\ &+ r^{p-1} \{1 + r^{(p-1)2\mu} + \ldots + r^{(p-1)^2\mu}\}.\end{aligned}$$

Every solution of the congruence $2k\mu + 1 \equiv 0 \pmod{p}$ gives a positive unit; k being a positive integer less than $\frac{1}{2}p$. Let θ_μ be the number of solutions for a given value of μ, then $\phi(\mu)$ contains $(p-1)\theta_\mu$ positive units and

$$\frac{(p-1)\{\frac{1}{2}(p+1)\} - (p-1)\theta_\mu}{p-1}$$

negative units, therefore

$$\phi(\mu) = p\theta_\mu - \tfrac{1}{2}(p+1),$$

therefore $$P = \Sigma_{\mu=1}^{\mu=p-1} \omega^{m \operatorname{ind} \mu} \{p\theta_\mu - \tfrac{1}{2}(p+1)\},$$

or, if m is not divisible by $p-1$,

$$P = p\Sigma_{\mu=1}^{\mu=p-1} \omega^{m \operatorname{ind} \mu} . \theta_\mu.$$

To every value of μ (less than p) corresponds one value of μ' (less than p), and one only, such that

$$\mu\mu' \equiv 1 \pmod{p},$$

therefore the number of solutions of $2k\mu + 1 \equiv 0 \pmod{p}$ is the same as the number of solutions of $2k + \mu' \equiv 0 \pmod{p}$.

Therefore

$$\theta_\mu = 1 \text{ if } \mu' \text{ be odd,}$$
$$= 0 \text{ if } \mu' \text{ be even.}$$

Hence
$$\theta_\mu + \theta_{p-\mu} = 1$$
$$\operatorname{ind}(p-\mu) = \operatorname{ind}\mu \pm \tfrac{1}{2}(p-1),$$
$$\operatorname{ind}\mu + \operatorname{ind}\mu' \equiv 0 \pmod{p-1}.$$

Therefore

$$P = p\Sigma_{\mu'=1}^{\mu'=p-1}\omega^{-m\operatorname{ind}\mu'}\theta_\mu$$
$$= p\left\{\Sigma_{\mu'=1}^{\mu'=\frac{1}{2}(p-1)}\omega^{-m\operatorname{ind}\mu'}\theta_\mu + (-1)^m\Sigma_{\mu'=1}^{\mu'=\frac{1}{2}(p-1)}\omega^{-m\operatorname{ind}\mu'}(1-\theta_\mu)\right\};$$

or, if m be odd,

$$P = p\Sigma_{\mu'=1}^{\mu'=\frac{1}{2}(p-1)}\omega^{-m\operatorname{ind}\mu'}(2\theta_\mu - 1) = p\Sigma_{\mu'=\frac{1}{2}(p+1)}^{\mu'=p-1}\omega^{-m\operatorname{ind}\mu'}(2\theta_\mu - 1).$$

VI. If $p = 4n+3$ and $m = \frac{1}{2}(p-1)$,

$$P = -2/p.ph,$$

and therefore

$$-2/p.ph = p\Sigma_{\mu'=1}^{\mu'=\frac{1}{2}(p-1)}\mu'/p\,(2\theta_\mu - 1) = p\Sigma_{\mu'=\frac{1}{2}(p+1)}^{\mu'=p-1}\mu'/p\,(2\theta_\mu - 1),$$

or

$$-2/p.h = -h + 2\Sigma_{\mu'=1}^{\mu'=\frac{1}{2}(p-1)}\mu'/p.\theta_\mu = h + 2\Sigma_{\mu'=\frac{1}{2}(p+1)}^{\mu'=p-1}\mu'/p.\theta_\mu.$$

From these two equations h may be expressed in terms of certain values of $\mu'/p\theta_\mu$, whilst another set of these values is found equal to 0.

Thus, if $p = 8n+3$, and therefore $2/p = -1$, the first equation gives

$$h = \Sigma_{\mu'=1}^{\mu'=\frac{1}{2}(p-1)}\mu'/p\theta_\mu$$

$= \Sigma\mu/p$ for odd values of μ between 1 and $\frac{1}{2}(p-1)$

$= -\Sigma\mu/p$ for even values of μ between $\frac{1}{2}(p+1)$ and $p-1$

$$= \Sigma_{\frac{1}{4}(p+1)}^{\frac{1}{2}(p-1)}\mu/p.$$

The second equation gives

$$\Sigma_1^{\frac{1}{4}(p-3)}\mu/p = 0.$$

Similarly, if $p = 8n + 7$,

$$h = \Sigma_1^{\frac{1}{4}(p-3)} \mu/p$$

and
$$\Sigma_{\frac{1}{4}(p+1)}^{\frac{1}{2}(p-1)} \mu/p = 0.$$

These results were obtained in the previous paper by a different method.

VII. From a consideration of the product

$$\left(\omega^m, \frac{1}{1+r^s}\right)(\omega^{-m}, r^t)$$

we get

$$-2/p.s/p.t/p.h = \Sigma_{\mu=1}^{\mu=p-1} \mu/p.\theta_\mu$$

$$= -h + 2\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \mu/p.\theta_\mu = h + 2\Sigma_{\mu=\frac{1}{2}(p+1)}^{\mu=\frac{1}{2}(p-1)} \mu/p.\theta_\mu,$$

where θ_μ is the number of solutions of the congruence $2ks + \mu t \equiv 0 \pmod p$: k being a positive integer less than $\frac{1}{2}p$.

VIII. $\left(\omega^m, \frac{1}{1+r}\right)$ may be easily connected with the resolvent function of Lagrange. For

$$\left(\omega^m, \frac{1}{1+r}\right) = \frac{1}{1+r} + \frac{\omega^m}{1+r^g} + \frac{\omega^{2m}}{1+r^{g^2}} + \ldots \frac{\omega^{(p-2)m}}{1+r^{g^{p-2}}}$$

$$= 1 + r^2 + r^4 + \ldots + r^{p-1}$$

$$+ \omega^m (1 + r^{2g} + r^{4g} + \ldots + r^{(p-1)g})$$

$$+$$

$$\vdots$$

$$+ \omega^{(p-2)m} (1 + r^{2g^{p-2}} + r^{4g^{p-2}} + \ldots + r^{(p-1)g^{p-2}})$$

$$= (\omega^m, 1) + (\omega^m, r^2) + (\omega^m, r^4) + \ldots + (\omega^m, r^{p-1})$$

$$= (\omega^m, 1) + (\omega^m, r)\,\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{-m \operatorname{ind} 2\mu}$$

$$= (\omega^m, 1) + (\omega^m, r)\,\omega^{-m \operatorname{ind} 2}\,\Sigma_{\mu=1}^{\mu=\frac{1}{2}(p-1)} \omega^{-m \operatorname{ind} \mu},$$

which shows that, if m be even and not divisible by $p-1$,

$$\left(\omega^m, \frac{1}{1+r}\right) = \left(\omega^{-m}, \frac{1}{1+r}\right) = 0.$$

IX. If $p=4n+3$ is prime, or the product of different primes,

$$\left(\omega^0, \frac{1}{1+r}\right)=\Sigma\frac{1}{1+r^\alpha}+\Sigma\frac{1}{1+r^\beta}=\tfrac{1}{2}(p-1),$$

$$\left(\omega^{\frac{1}{2}(p-1)}, \frac{1}{1+r}\right)=\Sigma\frac{1}{1+r^\alpha}-\Sigma\frac{1}{1+r^\beta}=2/p\,(\gamma_0-\gamma_1)\,h,$$

from which

$$4\Sigma\frac{1}{1+r^\alpha}\,\Sigma\frac{1}{1+r^\beta}=\{\tfrac{1}{2}(p-1)^2\}+ph^2.$$

Hence $\Sigma\dfrac{1}{1+r^\alpha}$ and $\Sigma\dfrac{1}{1+r^\beta}$ are the roots of the equation

$$x^2-\{\tfrac{1}{2}(p-1)x\}+\tfrac{1}{4}[\{\tfrac{1}{2}(p-1)\}^2+ph^2]=0,$$

or

$$\{2x-\tfrac{1}{2}(p-1)\}^2+ph^2=0.$$

A THEOREM IN COMPOUND DETERMINANTS.

By *Prof. E. J. Nanson.*

1. Let $(abcd)$ denote the determinant formed with the constituents a, a', a'', a''', b, b', &c., and let A, A', &c. denote the cofactors of a, a', &c. in $(abcd)$. Then since the determinant $(xbcd)$ found by replacing a's by x's has the value $Ax+A'x'+A''x''+A'''x'''$, it follows that

$$\begin{vmatrix} (xb_1c_1d_1) & (yb_1c_1d_1) & (zb_1c_1d_1) & (wb_1c_1d_1) \\ (a_2xc_2d_2) & (a_2yc_2d_2) & (a_2zc_2d_2) & (a_2wc_2d_2) \\ (a_3b_3xd_3) & (a_3b_3yd_3) & (a_3b_3zd_3) & (a_3b_3wd_3) \\ (a_4b_4c_4x) & (a_4b_4c_4y) & (a_4b_4c_4z) & (a_4b_4c_4w) \end{vmatrix}$$
$$=(xyzw)\,(A_1B_2C_3D_4)\,\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

From this identity several interesting results can be deduced, including the identities stated or proved by Dr. Muir, *Messenger*, Vol. XXXV., pp. 118–121.

2. In the first place, omitting the suffixes, we get

$$\begin{vmatrix} (xbcd) & (ybcd) & (zbcd) & (wbcd) \\ (axcd) & (aycd) & (azcd) & (awcd) \\ (abxd) & (abyd) & (abzd) & (abwd) \\ (abcx) & (abcy) & (abcz) & (abcw) \end{vmatrix}$$

$$= (xyzw)\,(ABCD) \;\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If in this we replace x, y, z, w by a, b, c, d, we fall back on Cauchy's theorem, viz.

$$(abcd)^3 = (ABCD),$$

so that the second member of (2) may be replaced by $(xyzw)\,(abcd)^3$. We thus get a theorem given by Vahlen, *Crelle*, CXII., pp. 306–310.

3. Again, in (1), replace x, y, z, w by a_4, b_4, c_4, d_4. Then in the last line of the determinant in the first member all the constituents vanish except the last, which has the value $(xyzw)$. Thus we deduce that

$$\begin{vmatrix} (a_4b_1c_1d_1) & (b_4b_1c_1d_1) & (c_4b_1c_1d_1) \\ (a_2a_4c_2d_2) & (a_2b_4c_2d_2) & (a_2c_4c_2d_2) \\ (a_3b_3a_4d_3) & (a_3b_3b_4d_3) & (a_3b_3c_4d_3) \end{vmatrix} = (A_1B_2C_3D_4),$$

and this is readily seen to be the identity formed by Dr. Muir on p. 121.

4. For the purpose of the next application it is convenient to re-state (1) in the form

$$\begin{vmatrix} (abcx) & (a_1b_1c_1x) & (a_2b_2c_2x) & (a_3b_3c_3x) \\ (abcy) & (a_1b_1c_1y) & (a_2b_2c_2y) & (a_3b_3c_3y) \\ (abcz) & (a_1b_1c_1z) & (a_2b_2c_2z) & (a_3b_3c_3z) \\ (abcw) & (a_1b_1c_1w) & (a_2b_2c_2w) & (a_3b_3c_3w) \end{vmatrix}$$

$$= (xyzw)\,(DD_1D_2D_3) \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

In this result for a_p, b_p, c_p write $\lambda_p a + \mu_p \alpha$, $\lambda_p b + \mu_p \beta$, $\lambda_p c + \mu_p \gamma$. Then if σ_p is an operator which applied to any expression containing a, b, c gives the sum of the results of

replacing p of the letters a, b, c by the corresponding letters in α, β, γ, so that

$$\sigma_1 f(abc) = f(\alpha bc) + f(a\beta c) + f(ab\gamma),$$
$$\sigma_2 f(abc) = f(a\beta\gamma) + f(\alpha b\gamma) + f(\alpha\beta c),$$
$$\sigma_3 f(abc) = f(\alpha\beta\gamma),$$

we deduce from (3), by equating the coefficients of $\lambda_1^2\mu_1\lambda_2\mu_2^2\mu_3^3$ that

$$\begin{vmatrix} (abcx) & \sigma_1(abcx) & \sigma_2(abcx) & \sigma_3(abcx) \\ (abcy) & \sigma_1(abcy) & \sigma_2(abcy) & \sigma_3(abcy) \\ (abcz) & \sigma_1(abcz) & \sigma_2(abcz) & \sigma_3(abcz) \\ (abcw) & \sigma_1(abcw) & \sigma_2(abcw) & \sigma_3(abcw) \end{vmatrix}$$
$$= (xyzw)\,(D,\ \sigma_1 D,\ \sigma_2 D,\ \sigma_3 D) \ldots\ldots\ldots\ldots(4),$$

where the second factor in the second member is the determinant whose first row is D, $\sigma_1 D$, $\sigma_2 D$, $\sigma_3 D$ and whose other rows are found by dashing the letter D as often as necessary.

5. If in (4) we replace x, y, z by α, β, γ, we find

$$\begin{vmatrix} (abc\alpha) & (a\beta c\alpha) + (ab\gamma\alpha) & (a\beta\gamma\alpha) \\ (abc\beta) & (abc\beta) + (ab\gamma\beta) & (ab\gamma\beta) \\ (abc\gamma) & (abc\gamma) + (a\beta c\gamma) & (a\beta c\gamma) \end{vmatrix} = (D,\ \sigma_1 D,\ \sigma_2 D,\ \sigma_3 D),$$

and this is the theorem enunciated without proof in Dr. Muir's paper. It is manifest that the method of proof here used applies to determinants of any order, although for brevity and clearness four line determinants have been used.

6. It is obvious from (3), or directly, by the composition of deficient arrays, that

$$\begin{vmatrix} (abx) & (a_1b_1x) & (a_2b_2x) & (a_3b_3x) \\ (aby) & (a_1b_1y) & (a_2b_2y) & (a_3b_3y) \\ (abz) & (a_1b_1z) & (a_2b_2z) & (a_3b_3z) \\ (abw) & (a_1b_1w) & (a_2b_2w) & (a_3b_3w) \end{vmatrix} = 0 \ldots\ldots(5).$$

The special interest of this result lies in the fact that it leads directly to a well-known result of Sylvester's.

Replacing a, b by y, z; a_1, b_1 by z, x; a_2, b_2 by x, y; a_4, b_4 by u, v, we get

$$\begin{vmatrix} (xyz) & . & . & (uvx) \\ . & (xyz) & . & (uvy) \\ . & . & (xyz) & (uvz) \\ (wyz) & (xwz) & (xyw) & (uvw) \end{vmatrix} = 0.$$

From this, by expansion and removal of the factor $(xyz)^2$, we get

$$(xyz)(uvw) = (wyz)(uvx) + (xwz)(uvy) + (xyw)(uvz),$$

and this is the result in question in the case of three line determinants.

7. The theorem (5) may be generalized. Let there be any number of letters a, b, c, &c., and consider the array

$$\left.\begin{matrix} a, & b, & c, & d, & \text{\&c.} \\ a', & b', & c', & d', & \text{\&c.} \\ & \text{\&c.} & & \text{\&c.} & \end{matrix}\right\} \text{..............(6)},$$

which has n rows. Let $\theta_1, \theta_2, \theta_3$, &c. be k-ads from a, b, c, &c., and let ϕ_1, ϕ_2, ϕ_3, &c. be $(n-k)$-ads from a, b, c, &c. With the k columns θ_p and the $n-k$ columns ϕ_q of the array (6) we can form a determinant of order n. Let this determinant be denoted by $(\theta_p\phi_q)$. Then the general theorem is that the compound determinant $|(\theta_p\phi_q)|$ vanishes, where $p, q = 1, 2, \ldots, \lambda$, provided λ is greater than n_κ the number of combinations of n letters κ together. From this theorem the most general case of Sylvester's Theorem, *Mathematical Papers* 39, can readily be deduced in the manner already indicated.

8. It is clear that consideration of the determinant $|(\theta_p\phi_q)|$ for the cases $\lambda \leqq n_\kappa$ will lead to wide extensions of the results in the first portion of this paper.

Melbourne.
March 28th, 1906.

ON THE SERIES $1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\&c.$

By *J. W. L. Glaisher*.

(SECOND PAPER.)

Introduction, § 1.

§ 1. IN a paper* at the beginning of Vol. XXXIII. of the *Messenger* I have given a list of a number of transformations of the series

$$1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\&c.$$

which I had then recently obtained. Since then I have been led in quite a different manner to other transformations of the same series, which are contained in a paper† published in Vol. XXXVII. of the *Quarterly Journal*. In the present paper I give, without proof, these transformations taken from that paper. The present paper is therefore a continuation of the one on the same subject in Vol. XXXIII. of the *Messenger*, and I have therefore given it the same title. The principal transformations of the series are given in §§ 2–14. Those quoted in §§ 15–19 are of much less interest. In the previous paper the series in question was denoted by u_2. In the paper in Vol. XXXVII. of the *Quarterly Journal* and in this paper it is denoted by g.

Transformation of g into series in which the numerators are unity and the denominators contain consecutive uneven numbers, §§ 2–14.

§ 2. In § 10 of the paper in Vol. XXXVII. of the *Quarterly Journal* (p. 334) it is shown that

$$\text{(i)}\quad g=\tfrac{5}{6}+2^2\left\{\frac{1}{1.3^2.5}-\frac{1}{3.5^2.7}+\frac{1}{5.7^2.9}-\&c.\right\},$$

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* "On the series $1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\&c.$," *Messenger*, Vol. XXXIII. (1903), pp. 1–19.

† "On the integral $\int_0 k^n K dk$," *Quarterly Journal*, Vol. XXXVII. (1906), pp. 329–349.

$$\text{(ii)}\quad g = \tfrac{569}{630} + 2^2.4^2 \left\{ \frac{1}{1.3.5^2.7.9} - \frac{1}{3.5.7^2.9.11} + \&c. \right\},$$

$$\text{(iii)}\quad g = \tfrac{854757}{935550} + 2^2.4^2.6^2 \left\{ \frac{1}{1.3.5.7^2.9.11.13} - \frac{1}{3.5.7.9^2.11.13.15} + \&c. \right\},$$

$$\text{(iv)}\quad g = \tfrac{8660207}{9459450} + 2^2.4^2.6^2.8^2 \left\{ \frac{1}{1.3.5.7.9^2.11.13.15.17} - \frac{1}{3.5.7.9.11^2.13.15.17.19} + \&c. \right\}.$$

These results are similar in character to a group of four formulæ which were given in the previous paper* in the *Messenger*, viz.,

$$(\alpha)\quad g = \tfrac{5}{4} - \tfrac{1}{2}\log 2 + \frac{1}{2.3^2.4} - \frac{1}{4.5^2.6} + \frac{1}{6.7^2.8} - \&c.,$$

$$(\beta)\quad g = \tfrac{25}{18} - \tfrac{2}{3}\log 2 - 2^2 \left\{ \frac{1}{1.2.3^2.4.5} - \frac{1}{3.4.5^2.6.7} + \&c. \right\},$$

$$(\gamma)\quad g = \tfrac{641}{450} - \tfrac{11}{15}\log 2 - 2^2.3^2 \left\{ \frac{1}{2.3.4.5^2.6.7.8} - \frac{1}{4.5.6.7^2.8.9.10} + \&c. \right\},$$

$$(\delta)\quad g = \tfrac{6367}{4410} - \tfrac{16}{21}\log 2 + 2^2.3^2.4^2 \left\{ \frac{1}{1.2.3\ 4.5^2.6.7.8.9} - \frac{1}{3.4.5.6.7^2.8.9.10.11} + \&c. \right\}.$$

In the second group consecutive numbers occur in the denominators; in the first group consecutive uneven numbers. In the second group, besides the series and a rational fraction, the quantity log 2 also occurs.

§3. Using m, as in the previous paper, to denote any uneven number, the two groups of formulæ may be written

$$\text{(i}'\text{)}\quad g = \tfrac{5}{6} + \Sigma_{m=3}^{m=\infty} (-1)^{\frac{1}{2}(m+1)} \frac{2^2}{m^2(m^2-2^2)},$$

* These formulæ, expressed in a slightly different form, are collected together on p. 17 of Vol. XXXIII. The formula (i) was given in the note on p. 6.

$$\text{(ii}'\text{)}\quad g = \tfrac{569}{630} + \Sigma_{m=5}^{m=\infty} (-1)^{\frac{1}{2}(m-1)} \frac{2^2.4^2}{m^2(m^2-2^2)(m^2-4^2)},$$

$$\text{(iii}'\text{)}\quad g = \tfrac{854757}{935550} + \Sigma_{m=7}^{m=\infty} (-1)^{\frac{1}{2}(m+1)} \frac{2^2.4^2.6^2}{m^2(m^2-2^2)(m^2-4^2)(m^2-6^2)},$$

$$\text{(iv}'\text{)}\quad g = \tfrac{8660207}{9459450} + \Sigma_{m=9}^{m=\infty} (-1)^{\frac{1}{2}(m-1)} \frac{2^2.4^2.6^2.8^2}{m^2(m^2-2^2)\ldots(m^2-8^2)},$$

and

$$(\alpha')\quad g = \tfrac{5}{4} - \tfrac{1}{2}\log 2 + \Sigma_{m=3}^{m=\infty} (-1)^{\frac{1}{2}(m+1)} \frac{1^2}{m^2(m^2-1^2)},$$

$$(\beta')\quad g = \tfrac{25}{18} - \tfrac{2}{3}\log 2 - \Sigma_{m=3}^{m=\infty} (-1)^{\frac{1}{2}(m+1)} \frac{1^2.2^2}{m^2(m^2-1^2)(m^2-2^2)},$$

$$(\gamma')\quad g = \tfrac{641}{450} - \tfrac{11}{15}\log 2 - \Sigma_{m=5}^{m=\infty} (-1)^{\frac{1}{2}(m-1)} \frac{1^2.2^2.3^2}{m^2(m^2-1^2)(m^2-2^2)(m^2-3^2)},$$

$$(\delta')\quad g = \tfrac{6367}{4410} - \tfrac{16}{21}\log 2 + \Sigma_{m=5}^{m=\infty} (-1)^{\frac{1}{2}(m-1)} \frac{1^2.2^2.3^2.4^2}{m^2(m^2-1^2)\ldots(m^2-4^2)}.$$

It may be remarked that (β') may be derived from (i′) and (α') by multiplying the latter by 2^2 and subtracting.

§4. The seven-place calculation of g from the formula (iii), including the factor $2^2.4^2.6^2$ in each term of the series, is

+ ·9136412	− ·0001263
24357	41
182	4
12	1
2	
	− ·0001309
+ ·9160965	
− ·0001309	
·9159656	

which is correct to the last figure.

The corresponding calculation from (γ) was given on p. 9 of Vol. XXXIII. Only four terms of the series in (γ) were required for the seven-place calculation, while seven terms of

the series in (iii) are required. The seven-place calculation of g from (iv) is

$$\begin{array}{rcr} +\cdot 9155085 & \qquad & -\cdot 0000205 \\ 4755 & & 5 \\ 25 & & \overline{} \\ 1 & & -\cdot 0000210 \\ \hline +\cdot 9159866 & & \\ -\cdot 0000210 & & \\ \hline \cdot 9159656 & & \end{array}$$

which also is correct to the last figure.

The corresponding seven-place calculation from (δ), for which only three terms of the series were required, was given on p. 15 of Vol. XXXIII.

§ 6. If we begin the series denoted by g with the term corresponding to the first term of the series into which it is transformed, the two groups of formulæ become

$$\text{(i}''\text{)} \quad \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \&\text{c.} = \tfrac{1}{6} - 2^2 \left\{\frac{1}{1.3^2.5} - \frac{1}{3.5^2.7} + \&\text{c.}\right\},$$

$$\text{(ii}''\text{)} \quad \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \&\text{c.} = \tfrac{1}{70} + 2^2.4^2 \left\{\frac{1}{1.3.5^2.7.9} - \frac{1}{3.5.7^2.9.11} + \&\text{c.}\right\},$$

$$\text{(iii}''\text{)} \quad \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{11^2} - \&\text{c.} = \tfrac{317}{20790} - 2^2.4^2.6^2 \left\{\frac{1}{1.3.5.7^2.9.11.13} - \&\text{c.}\right\},$$

$$\text{(iv}''\text{)} \quad \frac{1}{9^2} - \frac{1}{11^2} + \frac{1}{13^2} - \&\text{c.} = \tfrac{9497}{1351350}$$
$$+ 2^2.4^2.6^2.8^2 \left\{\frac{1}{1.3.5.7.9^2.11.13.15.17} - \&\text{c.}\right\},$$

and

$$(\alpha'') \quad \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \&\text{c.} = -\tfrac{1}{4} + \tfrac{1}{2}\log 2 - \left\{\frac{1}{2.3^2.4} - \frac{1}{4.5^2.6} + \&\text{c.}\right\},$$

$$(\beta'') \quad \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \&\text{c.} = -\tfrac{7}{18} + \tfrac{2}{3}\log 2$$
$$+ 2^2 \left\{\frac{1}{1.2.3^2.4.5} - \frac{1}{3.4.5^2.6.7} + \&\text{c.}\right\},$$

$$(\gamma'')\ \frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\&c.=\tfrac{241}{450}-\tfrac{11}{15}\log 2$$
$$-2^2.3^2\left\{\frac{1}{2.3.4.5^2.6.7.8}-\&c.\right\},$$

$$(\delta'')\ \frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{9^2}-\&c.=\tfrac{2447}{4410}-\tfrac{16}{21}\log 2$$
$$+2^2.3^2.4^2\left\{\frac{1}{1.2.3.4.5^2.6.7.8.9}-\&c.\right\}.$$

The first group are the formulæ (xvi)...(xix) of the paper in Vol. XXXVII. of the *Quarterly Journal* (p. 336). The second group were given on p. 18 of Vol. XXXIII. of the *Messenger*.

§ 7. The four formulæ (α)...(δ) were each obtained separately; but (i)...(iv) were given by a method which affords also the general formula* to which this group belongs, viz.,

$$(\text{v})\ g=\frac{A}{B}+2^2.4^2...(2r)^2\left\{\frac{1}{1.3...(2r-1)(2r+1)^2(2r+3)...(4r+1)}\right.$$
$$\left.-\frac{1}{3.5...(2r+1)(2r+3)^2(2r+5)...(4r+3)}+\&c.\right\},$$

where

$$\frac{A}{B}=2^2.4^2...(2r)^2\left\{\frac{1}{3^2.5^2...(2r-1)^2(2r+1)}\right.$$
$$+\frac{1}{3}\,\frac{1}{3^2.5^2...(2r-3)^2(2r-1)(2r+1)...(2r+3)}$$
$$\left.+\frac{1}{5}\,\frac{1}{3^2.5^2...(2r-5)^2(2r-3)...(2r+5)}+...+\frac{1}{2r-1}\,\frac{1}{3.5.7...(4r-1)}\right\}$$
$$-\frac{1}{2}\left\{1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+...+\frac{2^2.4^2...(2r-2)^2}{3^2.5^2...(2r-1)^2}\right\}.$$

§ 7. Expressed in the m-form this equation becomes

$$(\text{v}')\ g=\frac{A}{B}+(-1)^r\Sigma_{m=2r+1}^{m=\infty}(-1)^{\frac{1}{2}(m-1)}\frac{2^2.4^2...(2r)^2}{m^2(m^2-2^2)...\{m^2-(2r)^2\}},$$

---

* *Quarterly Journal*, Vol. XXXVII., p. 335.

where

$$\frac{A}{B}=(-1)^r\,\Sigma_{m=1}^{m=2r-1}(-1)^{\frac{1}{2}(m-1)}\frac{2^2.4^2\ldots(2r)^2}{m^2(m^2-2^2)\ldots\{m^2-(2r)^2\}}$$

$$-\frac{1}{2}\left\{1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+\ldots+\frac{2^2.4^2\ldots(2r-2)^2}{3^2.5^2\ldots(2r-1)^2}\right\}.$$

§ 8. It thus appears that the first of the two finite series which form the value of $\frac{A}{B}$ really consists of the first r terms of the infinite series in (v′), so that (v) may be written*

$$\text{(vi)}\ g=-\tfrac{1}{2}\left\{1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+\ldots+\frac{2^2.4^2\ldots(2r-2)^2}{3^2.5^2\ldots(2r-1)^2}\right\}$$

$$+(-1)^r\,\Sigma_{m=1}^{m=\infty}(-1)^{\frac{1}{2}(m-1)}\frac{2^2.4^2\ldots(2r)^2}{m^2(m^2-2^2)\ldots\{m^2-(2r)^2\}}.$$

From an algebraical point of view (vi) is a much simpler form of the result than (v), and there is no algebraical reason for separating the first r terms of the series from those which follow; but, regarded arithmetically as an equation connecting two numerical series, the form (v) is the natural one, as the first r terms of the series in (vi) when expressed in numbers are quite different in appearance from the terms in the infinite series in (v).

§ 9. Expressing the series defining g, (i), (ii), ... in their simplest algebraical form, we have

$$g=\Sigma_{m=1}^{m=\infty}(-1)^{\frac{1}{2}(m-1)}\frac{1}{m^2},$$

$$g=-\tfrac{1}{2}-\Sigma_{m=1}^{m=\infty}(-1)^{\frac{1}{2}(m-1)}\frac{2^2}{m^2(m^2-2^2)},$$

$$g=-\tfrac{1}{2}\left(1+\frac{2^2}{3^2}\right)+\Sigma_{m=1}^{m=\infty}(-1)^{\frac{1}{2}(m-1)}\frac{2^2.4^2}{m^2(m^2-2^2)(m^2-4^2)},$$

&c. &c.

§ 10. Taking the formula

$$g=\tfrac{10}{18}-32\,\Sigma_{m=3}^{m=\infty}(-1)^{\frac{1}{2}(m+1)}\frac{1}{m^2(m^2-2^2)^2},$$

---

* *Loc. cit.*, p. 337.

which was given on p. 4 of Vol. XXXIII., and beginning the series with $m=5$, it becomes

$$g=\tfrac{411}{450}+32\Sigma_{m=5}^{m=\infty}(-1)^{\frac{1}{2}(m-1)}\frac{1}{m^2(m^2-2^2)^2}\,.$$

Multiplying this equation by 2 and subtracting (ii′), we find, after reduction,

$$\text{(vii)}\quad g=\tfrac{2909}{3150}-768\Sigma_{m=5}^{m=\infty}(-1)^{\frac{1}{2}(m-1)}\frac{1}{m^2(m^2-2^2)^2(m^2-4^2)}\,,$$

or, giving to the first six terms their numerical values,

$$g=\tfrac{2909}{3150}-768\left\{\frac{1}{1.3^2.5^2.7^2.9}-\frac{1}{3.5^2.7^2.9^2.11}+\frac{1}{5.7^2.9^2.11^2.13}-\&c.\right\},$$

all the factors in a denominator being squared except the first and last.

The seven-place calculation of g from (vii) is

+ ·9234921	− ·0077400
2345	246
44	11
3	1
+ ·9237313	− ·0077658
− ·0077658	
·9159655	

differing from the true value by a unit in the last place.

§ 11. Beginning the series in (vii) with $m=7$, the formula becomes

$$g=\tfrac{181731}{198450}+768\Sigma_{m=7}^{m=\infty}\frac{1}{m^2(m^2-2^2)^2(m^2-4^2)}\,.$$

Multiplying this equation by 3 and subtracting (iii′), we find, after reduction,

$$\text{(viii)}\quad g=\tfrac{133423}{145530}-36864\Sigma_{m=7}^{m=\infty}\frac{1}{m^2(m^2-2^2)^2(m^2-4^2)(m^2-6^2)}\,,$$

or, giving to the first few terms their numerical values,

$$g=\tfrac{133423}{145530}-36864\left\{\frac{1}{1.3.5^2.7^4.9^4.11.13}-\frac{1}{3.5.7^4.9^2.11^2.13.15}+\&c.\right\},$$

in which the three middle factors in each denominator are squared.

The seven-place calculation of g from (viii) is

$+\cdot 9168075$	$-\cdot 0008660$
262	25
4	1
$+\cdot 9168341$	$-\cdot 0008686$
$-\cdot 0008686$	
$\cdot 9159655$	

differing from the true value by a unit in the last place.

§ 12. Similarly, by combining (viii) and (iv′), we may obtain a formula connecting g with the series

$$\Sigma_{m=9}^{m=\infty} \frac{1}{m^2(m^2-2^2)^2(m^2-4^2)(m^2-6^2)(m^2-8^2)},$$

and so on.

§ 13. If in (i), (ii)...(v) we combine into a single term each positive term and the following negative term, these formulæ become

$$g = \tfrac{5}{6} + 128\left\{\frac{1}{1.3^2.5^2.7} + \frac{2}{5.7^2.9^2.11} + \frac{3}{9.11^2.13^2.15} + \&c.\right\},$$

$$g = \tfrac{569}{630} + 1536\left\{\frac{3}{1.3.5^2.7^2.9.11} + \frac{5}{5.7.9^2.11^2.13.15} + \&c.\right\},$$

....,

and, in general, it is easy to show that

$$(-1)^r \Sigma_{m=2r+1}^{m=\infty} (-1)^{\frac{1}{2}(m-1)} \frac{1}{m^2(m^2-2^2)\ldots\{m^2-(2r)^2\}}$$

$$= 8(r+1)\, \Sigma_{n=r+1}^{n=\infty} \frac{n}{(4n^2-1)^2(4n^2-3^2)\ldots\{4n^2-(2r+1)^2\}},$$

where n has only even or only uneven values beginning with $n=r+1$ (*i.e.*, if $r+1$ is even, n is to have only even values, and if $r+1$ is uneven, n is to have only uneven values, $r+1$ being always the least value of n).

§ 14. It may be remarked that by combining (xi) of the previous paper (Vol. XXXIII., p. 8) with (vii) of this paper, we obtain a formula connecting g with the series

$$\Sigma_{m=5}^{m=\infty} \frac{1}{(m^2-4)^2 (m^2-16)(m^4+12m^2+64)}.$$

For, beginning the series with the term $m=5$, the formula (xi) of the previous paper becomes

$$g = \tfrac{20867}{22770} - 256\Sigma_{m=5}^{m=\infty} (-1)^{\frac{1}{2}(m-1)} \frac{1}{m^2(m^2-4)(m^4+12m^2+64)},$$

whence, by multiplying by 3 and subtracting (vii), we find, after reduction,

$$\text{(ix)}\quad g = \tfrac{66139}{72450} + 12288\,\Sigma_{m=5}^{m=\infty} \frac{1}{(m^2-4)^2(m^2-16)(m^4+12m^2+64)}.$$

The seven-place calculation of g from this formula is

+ ·9128916	− ·0000602
31304	5
42	
1	− ·0000607
+ ·9160263	
− ·0000607	
·9159656	

which is correct to the last figure.

§ 15. In § 25 (p. 342) of the paper in Vol. XXXVII. of the *Quarterly Journal* the three following formulæ are given for g:

$$g = \tfrac{1}{2} + \frac{1}{1^2.3} + \frac{2}{1.3^2.5} + \frac{2.4}{1.3.5^2.7} + \&c.,$$

$$g = \tfrac{11}{18} + 4\left\{\frac{1}{1^2.3.5} + \frac{2}{1.3^2.5.7} + \frac{2.4}{1.3.5^2.7.9} + \&c.\right\},$$

$$g = \tfrac{299}{450} + 24\left\{\frac{1}{1^2.3.5.7} + \frac{2}{1.3^2.5.7.9} + \&c.\right\}.$$

In the first series the last factor but one in the denominators is squared, in the second the last factor but two, and in the third the last factor but three. In all three series the factor which is squared exceeds the last factor in the numerator by unity.

It is shown in the paper how the other series of the same form may be obtained.

§ 16. The series in the preceding section converge very slowly and are unsuitable for the calculation of g. They may be regarded as affording transformations of the series g into other and more convergent series, or as affording the summation of the series on the right-hand side in terms of g. The latter point of view is perhaps the more natural.

Transformation of g into series having π as a factor,
§§ 17–20.

§ 17. In §§ 27–31 of the paper in Vol. XXXVII. of the *Quarterly Journal* (pp. 344–347) a number of formulæ in which g is expressed by means of series having π as a factor are given.

The first group of formulæ, which is derived from the elliptic-function expansion

$$\frac{2K}{\pi} = 1 + \frac{1^2}{2^2} k^2 + \frac{1^2.3^2}{2^2.4^2} k^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} k^6 + \&c.,$$

is

$$\text{(xiii)}\quad g = \frac{\pi}{4}\left\{1 + \frac{1^2}{2^2}\,\frac{1}{3} + \frac{1^2.3^2}{2^2.4^2}\,\frac{1}{5} + \frac{1^2.3^2.5^2}{2^2.4^2.6^2}\,\frac{1}{7} + \&c.\right\},$$

$$\text{(xiv)}\quad g = \tfrac{1}{6} + \tfrac{2}{3}\pi\left\{\frac{1}{1.3} + \frac{1^2}{2^2}\,\frac{1}{3.5} + \frac{1^2.3^2}{2^2.4^2}\,\frac{1}{5.7} + \&c.\right\},$$

$$\text{(xv)}\quad g = \tfrac{19}{82} + \tfrac{128}{41}\pi\left\{\frac{1}{1.3.5} + \frac{1^2}{2^2}\,\frac{1}{3.5.7} + \frac{1^2.3^2}{2^2.4^2.6^2}\,\frac{1}{5.7.9} + \&c.\right\},$$

$$\text{(xvi)}\quad g = \tfrac{713}{2646}$$

$$+ \tfrac{3072}{147}\pi\left\{\frac{1}{1.3.5.7} + \frac{1^2}{2^2}\,\frac{1}{3.5.7.9} + \frac{1^2.3^2}{2^2.4^2}\,\frac{1}{5.7.9.11} + \&c.\right\},$$

....

Similar formulæ may be obtained in which the factors which occur in the denominators of the fractions which multiply the quantities $1, \frac{1^2}{2^2}, \frac{1^2.3^2}{2^2.4^2}, \ldots$ are all increased by $2, 4, \ldots$ *ex. gr.* in which the series are

$$\frac{1}{3}+\frac{1^2}{2^2}\frac{1}{5}+\frac{1^2.3^2}{2^2.4^2}\frac{1}{7}+\&\text{c.}, \qquad \frac{1}{5}+\frac{1^2}{2^2}\frac{1}{7}+\&\text{c.},$$

$$\frac{1}{3.5}+\frac{1^2}{2^2}\frac{1}{5.7}+\&\text{c.}, \qquad \frac{1}{5.7}+\frac{1^2}{2^2}\frac{1}{7.9}+\&\text{c.},$$

&c. &c.

§ 18. The group derived from the formula

$$\frac{4V}{\pi}=1+\frac{1}{2^2}k^2+\frac{1^2}{2^2.4^2}k^4+\frac{1^2.3^2}{2^2.4^2.6^2}k^6+\&\text{c.}$$

is

$$\text{(xvii)}\quad g=-\tfrac{5}{2}+\pi\left\{1+\frac{1}{2^2}\frac{1}{3}+\frac{1^2}{2^2.4^2}\frac{1}{5}+\frac{1^2.3^2}{2^2.4^2.6^2}\frac{1}{7}+\&\text{c.}\right\},$$

$$\text{(xviii)}\quad g=-\tfrac{43}{30}+\tfrac{32}{15}\pi\left\{\frac{1}{1.3}+\frac{1}{2^2}\frac{1}{3.5}+\frac{1^2}{2^2.4^2}\frac{1}{5.7}+\&\text{c.}\right\},$$

$$\text{(xix)}\quad g=-\tfrac{1061}{1026}+\tfrac{512}{57}\pi\left\{\frac{1}{1.3.5}+\frac{1}{2^2}\frac{1}{3.5.7}+\frac{1}{2^2.4^2}\frac{1}{5.7.9}+\&\text{c.}\right\},$$

....,

and from the formula

$$\frac{2G}{\pi}=\tfrac{1}{2}k^2+\frac{1^2}{2^2.4}k^4+\frac{1^2.3^2}{2^2.4^2.6}k^6+\&\text{c.}$$

we may derive the group

$$\text{(xx)}\quad g=\tfrac{1}{2}+\frac{\pi}{4}\left\{\frac{1}{2}+\frac{1^2}{2^2.4}\frac{1}{3}+\frac{1^2.3^2}{2^2.4^2.6}\frac{1}{5}+\&\text{c.}\right\},$$

$$\text{(xxi)}\quad g=\tfrac{7}{10}+\tfrac{2}{5}\pi\left\{\frac{1}{2}\frac{1}{1.3}+\frac{1^2}{2^2.4}\frac{1}{3.5}+\frac{1^2.3^2}{2^2.4^2.6}\frac{1}{5.7}+\&\text{c.}\right\},$$

$$\text{(xxii)}\quad g=\tfrac{143}{186}+\tfrac{128}{93}\pi\left\{\frac{1}{2}\frac{1}{1.3.5}+\frac{1^2}{2^2.4}\frac{1}{3.5.7}+\&\text{c.}\right\}.$$

....

As in the case of the group in § 17 we may obtain results of the same kind involving similar series in which the factors occurring in the denominators of the fractions multiplying

$$1,\ \frac{1}{2^2},\ \frac{1}{2^2.4^2},\ \dots \text{ and } \frac{1}{2},\ \frac{1^2}{2^2.4},\ \frac{1^2.3^2}{2^2.4^2.6},\ \dots$$

are all increased by 2, 4, ...

§ 19. Some other formula in which the series is multiplied by π are also given in the same paper (§§ 348–349, pp. 32–34), *e.g.*

$$\text{(xxiii)}\ g = \tfrac{17}{18} - \tfrac{3}{16}\pi\left\{\frac{1}{2.3}\,\frac{1^2}{2.3} + \frac{1}{3.4}\,\frac{1^2.3^2}{2^2.4.5} + \frac{1}{4.5}\,\frac{1^2.3^2.5^2}{2^2.4^2.6.7} + \&c.\right\},$$

and

$$\text{(xxiv)}\ \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} - \&c.$$

$$= \frac{\pi}{4}\left\{\frac{1}{2}\,\frac{1^2}{2^2.3} + \frac{2}{3}\,\frac{1^2.3^2}{2^2.4^2.5} + \frac{3}{4}\,\frac{1^2.3^2.5^2}{2^2.4^2.6^2.7} + \&c.\right\},$$

$$\text{(xxv)}\ \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \&c.$$

$$= \frac{\pi}{4}\left\{\frac{1.2}{3.4}\,\frac{1^2.3^2}{2^2.4^2.5} + \frac{2.3}{4.5}\,\frac{1^2.3^2.5^2}{2^2.4^2.6^2.7} + \frac{3.4}{5.6}\,\frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8.9} + \&c.\right\},$$

$$\text{(xxvi)}\ \frac{1}{7^2} - \frac{1}{9^2} + \frac{1}{11^2} - \&c.$$

$$= \frac{\pi}{4}\left\{\frac{1.2.3}{4.5.6}\,\frac{1^2.3^2.5^2}{2^2.4^2.6^2.7} + \frac{2.3.4}{5.6.7}\,\frac{1^2.3^2.5^2.7^2}{2^2.4^2.6^2.8.9} + \&c.\right\}.$$

§ 20. The remark made in § 16 with respect to the formulæ of § 15 applies also to those in §§ 17–19, for, although the formulæ are so written as to express g in terms of a series, it is more natural to regard them as affording summations of these series in terms of the constant.

The formulæ in §§ 17–19, like those in § 15, converge very slowly, and are quite unsuitable for the calculation of g even to a small number of decimal places.

ON SETS OF INTERVALS IN A SIMPLY-ORDERED SERIES.

By *Philip E. B. Jourdain.*

In the present paper an attempt is made, firstly, to trace the development and generalisations of that theorem known by the names of Heine and Borel (§§ I.—V.); then to give its purely ordinal generalisation to simply-ordered series which are not necessarily series of real numbers (§§ III., VI.);* and, lastly, to point out, what does not seem to have been noticed hitherto, its connexion with the multiplicative axiom—an axiom which is usually, and unnecessarily in many cases, implied by current statements of the theorem in question (§§ III., IV., VII.).

I.

Let $F(x)$ be a real one-valued function of the real variable x, defined for every point of the interval $a \leqq x \leqq b$, where a and b are finite real numbers; and suppose that for $x = x_i$ it is "continuous," that is to say, given an arbitrarily small $\epsilon_i > 0$, we can find at least one $\delta_i > 0$ such that, for all values of x satisfying the condition

$$|x - x_i| < \delta_i,$$

we have $$|F(x) - F(x_i)| < \epsilon_i \quad \text{..................(1).}$$

There are, in fact, *many* such intervals δ_i, but there is an upper limit of the δ_i's for a fixed x_i and ϵ_i, such that (1) is fulfilled for any point x *within* the interval; let this upper limit be denoted by $\eta_i(x_i, \epsilon_i)$, or, more shortly, η_i. Thus, round every point x_i at which $F(x)$ is continuous, there is a defined interval η_i, of which x_i is the middle point.

We will suppose that $F(x)$ is continuous at *every* point of the interval $a \leqq x \leqq b$.† In this case it seems to have been Cantor who first saw the necessity for proving,‡ and proved, that, if we choose a number $\epsilon > 0$, as small as we wish, we

* On the separation out of purely ordinal theorems in mathematics, cf. my paper in *Crelle's Journal für Math.*, Bd. CXXVIII., 1905, pp. 170–171, 182–194.

† Of course, at $x = a$, $F(x)$ is only assumed to be continuous *on the right*; and only on the *left* at $x = b$.

‡ Cantor's theorem is necessary to make Cauchy's proof, that a continuous function is integrable ["Résumé des leçons" (1823), leçon 21; *Œuvres* (2), t. IV.], valid.

can find a number $\delta > 0$ such that, *wherever* x' *and* x'' *may be in the interval*, provided only that

$$|x' - x''| < \delta,$$

we shall have
$$|F(x') - F(x'')| < \epsilon.$$

This property is called the *continuity of* $F(x)$ *throughout the interval*, or (better) the *uniform continuity** of $F(x)$, and the proof was published by Heine.†

We choose, for each point x_i of $a \leqq x \leqq b$, that interval η_i referred to above, where $\epsilon_i = \epsilon$. For δ we take the lower limit of these η's, and thus the nerve of the proof consists in showing that $\delta > 0$. Now, if δ *could* be zero, there would be an infinite series of points

$$x_1, x_2, \ldots, x_\nu, \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

such that the series

$$\eta_1 > \eta_2 > \ldots > \eta_\nu > \ldots$$

converges to zero. If, then, x_ω is a point of condensation of (2), a finite η_ω belongs to it, and this is readily seen to be in contradiction with the preceding deduction that in any neighbourhood of x_ω there are points x_ν for which the corresponding η_ν's remain below any assignable length.‡

II.

Now, the theorem which underlies the above proof is not essentially connected with uniform continuity, and runs:

If round every point x of the interval $a \leqq x \leqq b$ there is a containing interval (some of which latter intervals must therefore overlap), there can be chosen from this aggregate of intervals a *finite* aggregate such that every x of $(a \ldots b)$ is within at least one. For, if an aggregate of intervals possessing this property were necessarily infinite, the aggregate

* See Dini's work of 1878, translated into German by Lüroth and Schepp under the title "Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse" (Leipzig, 1892), pp. 61—63. Dini's remark, on p. 63, that there is no object in Heine's distinction between uniformly and non-uniformly continuous functions is, apparently, due to a misunderstanding of Heine's meaning. Heine merely wished (like Dini) to point out that, though *à priori* one might think non-uniform continuity conceivable, it is in reality not so.

† "Die Elemente der Functionenlehre," Crelle's *Journ. für Math.*, Bd. LXXIV., 1872, p. 188.

‡ For other proofs see Dini—Lüroth, *op. cit.*, pp. 63—65; Lüroth, "Bemerkung über gleichmässige Stetigkeit," *Math. Ann.*, Bd. VI., 1873, pp. 319—320; Darboux, "Mémoire sur les fonctions discontinues," *Ann. de l' Ec. Norm.* (2), t. IV., p. 73.

of the left-end-points (for example) of the intervals would have a point of condensation, and this need never occur, since the point of condensation is itself within a finite interval which may replace an infinity of the condensing intervals.

If we suppose the aggregate of the intervals first mentioned in the above theorem to be *enumerable*, we have Borel's* theorem, and his proofs make use of this enumerability. However, this restriction is easily seen to be unnecessary,† and I give here a modification, of which the importance is explained at the end of § III., of Schönflies'‡ proof, which involves transfinite ordinal numbers of Cantor's second class.

III.

Let δ_1 be the special interval which contains b, the extreme right of the continuum $a \leqq x \leqq b$ as its internal point. There may be many intervals thus containing b, but we suppose, for definiteness, one special interval belongs to each x, just as in the case of uniform continuity, to each x_ι, one particular η_ν belonged. Further, we will denote b by x_1. Let x_2 be the left-end-point of δ_1, δ_2, the interval belonging to x_2, x_3 the left-end-point of δ_2, and so on. We thus obtain a series

$$x_1, x_2, x_3, \ldots, x_\nu, \ldots \qquad \ldots\ldots\ldots\ldots\ldots (3),$$

and we either reach (or surpass) a, or the points of (3) condense at a point to the right of a which is properly represented by x_ω, ω being the first transfinite ordinal number. Let δ_ω be the interval belonging to x_ω, and denote the left-end-point of δ_ω by $x_{\omega+1}$, and proceed as before, forming x's whose suffixes are numbers of Cantor's second class:

$$\omega, \omega+1, \ldots, \omega+\nu, \ldots \omega.2, \ldots \omega.\nu, \ldots \omega^2, \ldots \omega^\omega, \ldots \alpha, \ldots .$$

Now we must reach (or surpass) a with some x_α, where α is a member of the second class; for Cantor§ has proved that the aggregate of intervals on a straight line, any two of

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* Sur quelques points de la théorie des fonctions," *Ann de l'Ec. Norm.* (3), t. XII., 1895, p. 51; "Leçons sur la théorie des fonctions," Paris, 1898, pp. 42—43.

† See Borel's remark in "Leçons sur les fonctions de variables réelles," Paris, 1905, pp. 9—10; after Lebesgue, "Leçons sur l'intégration et la recherche des fonctions perimitives," Paris, 1904, pp. 62—63. Cf. the proofs of uniform continuity in the above works respectively, pp. 27—28 and pp. 21—22.

‡ "Die Entwickelung der Lehre von den Punktmannigfaltigkeiten," Leipzig, 1900, pp. 51—52.

§ Cantor, *Math. Ann.*, Bd. XX., 1882, p. 117; Schönflies, *op. cit.*, p. 13. The theorem is, indeed, more general, and applies to continua of more than one dimension.

which have at most one point in common, is either finite or enumerable; that is to say, is of cardinal number

$$\aleph_0$$

at most. Now the series of x's found defines such an aggregate, and, if this series did not break off with some suffix x of the second class, we should be able to define an aggregate of separated intervals on $a \leqq x \leqq b$ of cardinal number

$$\aleph_1,$$

a palpable contradiction.

We now replace this aggregate

$$\delta_1, \delta_2, \ldots, \delta_\omega, \delta_{\omega+1}, \ldots, \delta_a \quad \ldots\ldots\ldots\ldots\ldots(4)$$

by a finite aggregate as follows: In the series (4) δ_ω will contain all those points x_ν after some finite ν, and thus the finite series

$$\delta_1, \delta_2, \ldots, \delta_\nu, \delta_\omega$$

can replace, for the purposes of the theorem, the infinite series

$$\delta_1, \delta_2, \ldots, \delta_\nu, \ldots, \delta_\omega.$$

We can imagine this process carried out in succession (there is but one way of proceeding) for all those intervals δ_γ of which γ is a Limes number, and we thus get a finite series of δ's as required.

There are two very important points to be noticed in this proof. One is that what is essential in our use of Cantor's theorem on the enumerability of non-overlapping intervals is that it enables us to prove that a *is* reached (or surpassed) by *some* ordinal number of steps; it so happens that this number is of the *second* class, but that is unessential. In § VI. we shall have to consider aggregates in which ordinal numbers of higher classes are used, but we shall always be able to replace the series of intervals so defined by a *finite* series.

The second point is that the above process is unique throughout. If there were not a special δ associated with each x, as we supposed, but we had to choose one of several δ's for each x,* we should, in general, have an 'infinite

* This is the case with Schönflies (*op. cit.*, p. 51). In the enunciation of this 'Heine-Borel' theorem he merely says: "Jeder Punkt des Intervalls $a \ldots b$ innerer Punkt mindestens eines Intervalles δ ist," but in the proof: "Sei nämlich a_1 ein beliebiger Punkt und δ_1 *das* zugehörige Intervall" (italics mine). This is illegitimate, since there is no '*the*' corresponding interval if there are many. If we substitute (as we must) 'any' for 'the,' the multiplication axiom enters.

sequence of arbitrary choices' which implies an axiom (the "multiplicative axiom" of § VII. below) for its justification. Also, if the intervals lack their end-points,* there must evidently be freedom of choice of the x at each step, and thus, in general, the multiplicative axiom is again involved.

IV.

In the application of the theorem in question (known as the Heine-Borel theorem) to the proof of the theorem that every one-valued analytic function of a complex variable, all of whose singularities are non-essential, takes any assigned real or complex value at least once,† I have emphasised the difference that there is between the applications to this theorem and the theorem on uniform continuity.‡ In fact, in the proof (cf. above, § I.) of the theorem on uniform continuity, the transfinite numbers will only appear if the successive points x_ν of (2) are taken *within* the corresponding intervals η_ν, and then, in general, they will appear. This comes to supposing that the η's are intervals which are *open at the ends*. In the generalisation of the fundamental theorem of algebra referred to above, these intervals η are essentially open. And, as I have remarked above, the validity of proofs in which open intervals are used requires the multiplicative axiom. This axiom is dealt with further in § VII. below; in the next sections we will consider the various generalisations of the Heine-Borel theorem to any closed aggregate of real numbers (§ V.); to any simply ordered series considered in itself, and not as situated in another series (the continuum of real numbers, for example); and, lastly, to any simply ordered series situated in another series (for example, that formed by closing the original series) (§ VI.).

V.

Schönflies§ pointed out that the proof of the theorem on uniform continuity rests on the fact that the argument aggregate (of real numbers) for which the continuous function is defined is *closed*, and that the theorem in question does not hold for unclosed aggregates; but Veblen‖ seems to have been

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* As in the statement of the Heine-Borel theorem by Veblen (*Bull. Amer. Math. Soc.* (2), Vol. x., 1904, pp. 436–437).

† "On Functions, all of whose Singularities are Non-Essential," *Messenger of Math.*, 1901, pp. 166–171.

‡ *Loc. cit*, pp. 166–167, 170–171.

§ *Op. cit.*, p. 119.

‖ "The Heine-Borel Theorem," *Bull. Amer. Math. Soc.* (2), Vol. x., 1904, pp. 436–439.

the first to prove explicitly that the Heine-Borel theorem holds for any closed aggregate of real numbers, and only for such aggregates.* In fact, in an aggregate $c < x \leqq b$, in which every point x was within an interval δ, and c is to the left of every δ, it is evident that, in the process of § III., we come upon an infinite series (2) condensing at c, and the corresponding δ's, of which, since ϵ has no corresponding δ_c, no finite part, together with δ_c, can be chosen so as to satisfy our requirements. The Heine-Borel theorem cannot, then, be applied to unclosed aggregates of real numbers; that it can be applied to closed aggregates is quite evident. Here the aggregate is contained in the aggregate also of real numbers; but, if we consider a series in itself (that is to say, if we consider a series merely as an ordered aggregate, without implying that it is contained in another), we shall find that 'closure' (as applied to *types*) is required.

VI.

A simply-ordered series M is said† to be 'closed' when every 'fundamental series' in M, that is to say, every series of type

$$\omega \text{ or } {}^*\omega$$

of elements of M in the order in which they occur in M, has a Limes.‡ More fully, suppose that $\{a_\nu\}$ is a series of elements in M of type ω; if M is 'closed' there is an element a_ω of M such that a_ν precedes a_ω, where ν is any finite ordinal number, and, for every element m preceding a_ω, there is a finite ordinal number ν_0 such that m precedes a_{ν_0}; and analogously for any series of type

$${}^*\omega.$$

It tends to greater clearness to say, for this, that *the type* of M is closed. For a series of real numbers may not be 'closed,' in the sense of having the points of condensation as members, although its type is§; thus the series

$$2, \tfrac{3}{2}, \tfrac{5}{4}, \ldots, \frac{2\nu+1}{2\nu}, \ldots, 0,$$

regarded as a series situated in the number-continuum, is not

* W. H. Young ("Overlapping Intervals," *Proc. Lond. Math. Soc.*, Vol. XXXV., 1903, p. 384) proved that to any closed linear aggregate the Heine-Borel theorem is applicable.

† Cf. Cantor, *Math. Ann.*, Bd. XLVI., 1895, p.

‡ For this word ("Grenzelement"), cf. *Phil. Mag.*, March, 1904, p. 296, and Crelle's *Journ. für Math.*, Bd. CXXVIII., 1905, p. 187.

§ Cf. Schönflies, *op. cit.*, pp. 63–64, 28, 31–33.

closed, although its type, $\omega+1$, is. In general, the type of a series may be closed, though the series may not be closed with respect to another series containing it.

Consider any series with ends; suppose that each term x of it is contained within some particular 'interval' of terms of the series between ξ and ζ (the 'interval' being the series of these latter terms, together with ξ and ζ),* and apply the argument of § III. If we suppose that we can get from end to end of the series by using the ordinal numbers less than γ, where γ may be of any class, we can replace all the intervals by a finite partial-series of them, by the successive abolition of Limes-terms.

Thus we see that, given a series of type β, the necessary and sufficient condition that the Heine-Borel theorem should should hold of the series is that β is closed. Also, if the series is contained in another (or the same) closed one, just as a 'point'-aggregate is contained in the continuum, the necessary and sufficient condition that the Heine-Borel theorem should be applicable to the former is that it should be closed with respect to the latter; that is to say, the Limes of fundamental series of terms of the former in the latter is also a term of the former. We must note that a series of closed type may be contained in a series of closed type, and yet the former may not be closed with respect to the latter; thus the series

$$2,\ \tfrac{3}{2},\ \tfrac{5}{4},\ \ldots,\ \frac{2\nu+1}{2\nu},\ \ldots,\ 0,$$

is of closed type $\omega+1$, and is contained in the series

$$2,\ \tfrac{3}{2},\ \tfrac{5}{4},\ \ldots,\ \frac{2\nu+1}{2\nu},\ \ldots,\ \tfrac{1}{2},\ 0,$$

of closed type $\omega+2$, and yet the former is not closed with respect to the latter, for the Limes of the latter is $\frac{1}{2}$, which is not a member of the former. In this case neither series, when considered as a series of real numbers, is closed.

In connexion with this ordinal generalisation, a reference may be permitted to the method I used† in a purely ordinal theorem in the theory of functions. With a series containing between any two terms a term of a series of type η, a purely ordinal analogue of the well-known Bolzano-Weierstrassian process exists.

* Of course, the ends of the series are exceptions to this; they are ends of certain intervals.

† Crelle's *Journ. für Math.*, Bd. CXXVII., 1905, p. 191.

Now, just as this process may be replaced by the Heine-Borel theorem when we are dealing with an aggregate of real numbers,* so the above analogue may be replaced by our theorem on sets of intervals in a series contained in one whose type is closed by considering the consequence of supposing the series to be infinite and yet to have no Limes.

VII.

We shall now discuss briefly the "multiplicative axiom."† This axiom, which is necessary for the validity of the definition of the product of an infinity of cardinal numbers, asserts that, if k is a class of classes no two of which have any common terms, the "multiplicative class of k" (the class formed by picking one, and only one, term out of each class belonging to k, and doing this in all possible ways) has at least one term: that is to say, there is at least one rule by which we can pick out one term of each member of k. An analogous axiom is that of Zermelo, which a proof that a class u can be well-ordered involves, and which was stated by Zermelo (in essentials) as follows: If k is the class of all classes (none of which are null) contained in u, there is a many-one relation R, whose domain is k, which is such that, if x is a member of k, the term to which x has the relation R is a member of x.

Now we may observe that Zermelo's axiom can easily be proved to be equivalent to the above "multiplicative axiom," in which the words: 'no two of which have any common terms' are left out. I prefer to call the *resulting* statement the "multiplicative axiom," so that, in this sense, the multiplicative axiom is equivalent to Zermelo's, while the axiom which is necessary for the validity of the conception of the product of an infinity of cardinal numbers is a particular case of it.

Lastly, it is not necessary to assume that there *is* such a thing as a multiplicative *class* (we know, from other arguments, that certain propositional functions do not determine classes), and it is better to state the "multiplicative axiom" as follows:

Let k be any class of classes, then the complex of statements: v is a class; v is contained in the 'logical sum of k'‡; if x is

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* Veblen, *loc. cit.*, p. 437.

† Cf. Russell, *Proc. Lond. Math. Soc.* (2), Vol. IV., pp. 47–53; and Hardy, *ibid* pp 14–17.

‡ That is to say, the class of those terms z such that the complex of propositions 'x is a k' and 'z is an x' is not false.

any member of k, then x and v have one, and only one, term in common; is not false for all v's.

Referring again to §§ III. and IV. above, we see that the validity of Schonflies' and Veblen's statements of the Heine-Borel theorem, and my own proof of the fundamental theorem of Algebra, assume the multiplicative axiom, while the assumption is not essential and is not necessary for the theorem on the uniform continuity of a function. I do not mean to imply that there is any doubt about the truth of the multiplicative axiom; to admit this would seem to throw doubt on the natural extension of one's imagination from finite to transfinite series of operations, against which no reason has been urged; but it is as well to keep our assumption distinct.

The Manor House,
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April 21st, 1906.

ON VARIOUS EXPRESSIONS FOR h, THE NUMBER OF PROPERLY PRIMITIVE CLASSES FOR A NEGATIVE DETERMINANT.

(FOURTH PAPER.)

By *H. Holden*.

1. USING the notation

$$\frac{x^p - 1}{x - 1} = S^2 - (-1)^{\frac{1}{2}(p-1)} px T^2,*$$

$$H = \frac{h}{2 - (2/p)},$$

the chief results obtained are:—

(a) for $p = 4n + 3$,

$$\left(\frac{dS}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)} \frac{pH\{1 - (2/p)\}}{2},$$

$$\left(\frac{dT}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)} \frac{p-3}{4},$$

$$\left(\frac{d^2S}{dx^2}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)} \frac{p(p-3)H\{1-(2/p)\}}{4},$$

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* A proof of this transformation is given in the *Quarterly Journal*, Vol. XXXIV., pp. 235–240.

$$\left(\frac{d^2T}{dx^2}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)}\left[\frac{5p^2-30p+61}{48} - \frac{pH^2\{1-(2/p)\}}{2}\right];$$

(*b*) for $p = 4n+1$,

$$\left(\frac{dS}{dx}\right)_{x=-1} = -(2/p)\frac{p-1}{4},$$

$$\left(\frac{dT}{dx}\right)_{x=-1} = -(2/p)\frac{h}{2},$$

$$\left(\frac{d^2S}{dx^2}\right)_{x=-1} = +(2/p)\left\{\frac{3(p-1)^2}{16} - \frac{ph^2}{4}\right\},$$

$$\left(\frac{d^2T}{dx^2}\right)_{x=-1} = +(2/p)\frac{(p-5)h}{4}.$$

2. Let $p = 4n+3$, and put

$$X_1 = \Pi(x^{\frac{1}{2}} + r^{\alpha}).\Pi(x^{\frac{1}{2}} - r^{\beta}) = S + i\sqrt{p}.x^{\frac{1}{2}}T,$$

$$X_2 = \Pi(x^{\frac{1}{2}} - r^{\alpha}).\Pi(x^{\frac{1}{2}} + r^{\beta}) = S - i\sqrt{p}.x^{\frac{1}{2}}T,$$

where α and β are the quadratic residues and non-residues of p, less than p.

Then $$2S = X_1 + X_2,$$

and by logarithmic differentiation

$$\frac{1}{X_1}\frac{dX_1}{dx} = \Sigma\frac{\frac{1}{2}x^{-\frac{1}{2}}}{x^{\frac{1}{2}}+r^{\alpha}} + \Sigma\frac{\frac{1}{2}x^{-\frac{1}{2}}}{x^{\frac{1}{2}}-r^{\beta}},$$

$$\frac{dX_1}{dx} = \frac{X_1}{2}\left\{\Sigma\frac{1}{x+x^{\frac{1}{2}}r^{\alpha}} + \Sigma\frac{1}{x-x^{\frac{1}{2}}r^{\beta}}\right\},$$

and similarly for $\dfrac{dX_2}{dx}$.

Then, as

$$(X_1)_{x=1} = -(X_2)_{x=1} = -(-1)^{\frac{1}{2}(h+1)}(2/p)\,i\sqrt{p}.(2/p)^*$$
$$= -(-1)^{\frac{1}{2}(h+1)}i\sqrt{p},$$

we have

$$2\left(\frac{dS}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h+1)}\frac{i\sqrt{p}}{2}\left\{\Sigma\frac{1}{1-r^{\alpha}} + \Sigma\frac{1}{1+r^{\beta}} - \Sigma\frac{1}{1+r^{\alpha}} - \Sigma\frac{1}{1-r^{\beta}}\right\}$$

$$= (-1)^{\frac{1}{2}(h+1)}\frac{i\sqrt{p}}{2}\{i\sqrt{p}.H - (2/p)\,i\sqrt{p}.h\};$$

* The proofs of this, and other results, assumed in the present paper, may be found in the first paper of this series. See *Messenger of Mathematics*, New Series Vol. XXXV., pp. 73–80.

therefore $$\left(\frac{dS}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)} \cdot \frac{pH\{1-(2/p)\}}{2}.$$

3. From $$\frac{x^p - 1}{x - 1} = S^2 + pxT^2,$$

we get $$\frac{d}{dx}\left(\frac{x^p-1}{x-1}\right) = 2S\frac{dS}{dx} + pT^2 + 2pxT\frac{dT}{dx},$$

But $$2(S)_{x=1} = (X_1)_{x=1} + (X_2)_{x=1} = 0,$$

and $$2i\sqrt{p}\,(T)_{x=1} = (X_1)_{x=1} - (X_2)_{x=1}$$
$$= 2(-1)^{\frac{1}{2}(h-1)}\, i\sqrt{p}.$$

Therefore $$(T)_{x=1} = (-1)^{\frac{1}{2}(h-1)},$$

and so
$$\frac{p(p-1)}{2} = 0 + p + 2p(-1)^{\frac{1}{2}(h-1)}\left(\frac{dT}{dx}\right)_{x=1},$$

or $$\left(\frac{dT}{dx}\right)_{x=1} = (-1)^{\frac{1}{2}(h-1)}\,\frac{p-3}{4}.$$

4. We have $$2\frac{d^2S}{dx^2} = \frac{d^2X_1}{dx^2} + \frac{d^2X_2}{dx^2}.$$

Now $$\frac{dX_1}{dx} = \frac{X_1}{2}\left(\Sigma\frac{1}{x + x^{\frac{1}{2}}r^{\alpha}} + \Sigma\frac{1}{x - x^{\frac{1}{2}}r^{\beta}}\right),$$

therefore
$$\frac{d^2X_1}{dx^2} = \frac{1}{2}\frac{dX_1}{dx}\left(\Sigma\frac{1}{x + x^{\frac{1}{2}}r^{\alpha}} + \Sigma\frac{1}{x - x^{\frac{1}{2}}r^{\beta}}\right)$$
$$- \frac{X_1}{2}\left\{\Sigma\frac{1 + \frac{1}{2}x^{-\frac{1}{2}}r^{\alpha}}{(x + x^{\frac{1}{2}}r^{\alpha})^2} + \Sigma\frac{1 - \frac{1}{2}x^{-\frac{1}{2}}r^{\beta}}{(x - x^{\frac{1}{2}}r^{\beta})^2}\right\}$$
$$= \frac{X_1}{4}\left(\Sigma\frac{1}{x + x^{\frac{1}{2}}r^{\alpha}} + \Sigma\frac{1}{x - x^{\frac{1}{2}}r^{\beta}}\right)^2$$
$$- \frac{X_1}{2}\left\{\Sigma\frac{1 + \frac{1}{2}x^{-\frac{1}{2}}r^{\alpha}}{(x + x^{\frac{1}{2}}r^{\alpha})^2} + \Sigma\frac{1 - \frac{1}{2}x^{-\frac{1}{2}}r^{\beta}}{(x - x^{\frac{1}{2}}r^{\beta})^2}\right\},$$

with a similar expression for $\frac{d^2X_2}{dx^2}$.

As $(X_2)_{x=1} = -(X_1)_{x=1}$, we have

$$2\left(\frac{d^2S}{dx^2}\right)_{x=1} = \left(\frac{X_1}{4}\right)_{x=1}\left\{\left(\Sigma\frac{1}{1+r^\alpha}+\Sigma\frac{1}{1-r^\beta}\right)^2-\left(\Sigma\frac{1}{1-r^\alpha}+\Sigma\frac{1}{1+r^\beta}\right)^2\right\}$$

$$-\left(\frac{X_1}{2}\right)_{x=1}\left\{\Sigma\frac{1+\frac{1}{2}r^\alpha}{(1+r^\alpha)^2}+\Sigma\frac{1-\frac{1}{2}r^\beta}{(1-r^\beta)^2}-\Sigma\frac{1-\frac{1}{2}r^\alpha}{(1-r^\alpha)^2}-\Sigma\frac{1+\frac{1}{2}r^\beta}{(1+r^\beta)^2}\right\}.$$

The coefficient of $(\frac{1}{4}X_1)_{x=1}$ is equal to

$$(p-1)\{(2/p)\,i\sqrt{p}.h - i\sqrt{p}.H\}.$$

Again, as

$$\Sigma\frac{1+\frac{1}{2}r^\alpha}{(1+r^\alpha)^2}-\Sigma\frac{1+\frac{1}{2}r^\beta}{(1+r^\beta)^2}=\Sigma\left\{\frac{1+\frac{1}{2}r^\alpha}{(1+r^\alpha)^2}-\frac{1+\frac{1}{2}r^{-\alpha}}{(1+r^{-\alpha})^2}\right\}$$

$$=\Sigma\frac{1-r^\alpha}{1+r^\alpha} \text{ by ordinary reduction}$$

$$=\Sigma\left\{\frac{1}{1+r^\alpha}-\frac{1}{1+r^{-\alpha}}\right\}$$

$$=\Sigma\frac{1}{1+r^\alpha}-\Sigma\frac{1}{1+r^\beta},$$

and similarly

$$\Sigma\frac{1-\frac{1}{2}r^\alpha}{(1-r^\alpha)^2}-\Sigma\frac{1-\frac{1}{2}r^\beta}{(1-r^\beta)^2}=\Sigma\frac{1}{1-r^\alpha}-\Sigma\frac{1}{1-r^\beta},$$

the coefficient of $-(\frac{1}{2}X_1)_{x=1}$ is $(2/p)\,i\sqrt{p}.h - i\sqrt{p}.H$, and so

$$2\left(\frac{d^2S}{dx^2}\right)_{x=1}=(\tfrac{1}{2}X_1)_{x=1}\left[\tfrac{1}{2}(p-1).\{(2/p)i\sqrt{p}.h-i\sqrt{p}.H\}\right.$$

$$\left.-\{(2/p)\,i\sqrt{p}.h-i\sqrt{p}.H\}\right]$$

$$=(-1)^{\frac{1}{2}(h+1)}.\tfrac{1}{2}(i\sqrt{p})\left[\tfrac{1}{2}(p-3).\{(2/p)\,i\sqrt{p}.h-i\sqrt{p}.H\}\right],$$

therefore $\left(\frac{d^2S}{dx^2}\right)_{x=1}=(-1)^{\frac{1}{2}(h-1)}\frac{p.(p-3)H\{1-(2/p)\}}{4}.$

This result shows that when $p=8n+7$, $(x-1)^3$ is a factor of S. By differentiating twice the equation $\frac{x^p-1}{x-1}=S^2+pxT^2$, putting $x=1$, and substituting known values, we get

$$\left(\frac{d^2T}{dx^2}\right)_{x=1}=(-1)^{\frac{1}{2}(h-1)}\left[\frac{5p^2-30p+61}{48}-\frac{pH^2\{1-(2/p)\}}{2}\right].$$

5. Let $p = 4n + 1$,

and $X_1 = \Pi (x^{\frac{1}{2}} + r^\alpha) \, \Pi (x^{\frac{1}{2}} - r^\beta) = S + \sqrt{p} . x^{\frac{1}{2}} T$,

$$X_2 = \Pi (x^{\frac{1}{2}} - r^\alpha) \, \Pi (x^{\frac{1}{2}} + r^\beta) = S - \sqrt{p} . x^{\frac{1}{2}} T.$$

Now

$$\Pi (i + r^\alpha) \, \Pi (i - r^\beta) = \Pi (i + r^\alpha)(i + r^{-\alpha}) \, \Pi (i - r^\beta)(i - r^{-\beta}),$$

where α and β are now less than $\frac{1}{2}p$,

$$= \Pi i (r^\alpha + r^{-\alpha}) . \Pi (-i)(r^\beta + r^{-\beta})$$
$$= \Pi (r^n + r^{-n}),$$

where n is any positive integer less than $\frac{1}{2}p$.

Similarly

$$\Pi (i - r^\alpha) \, \Pi (i + r^\beta) = \Pi (r^n + r^{-n}).$$

But $\Pi (r^n + r^{-n}) = \Pi . 2 \cos \frac{2n\pi}{p}$, and therefore the sign of this product is $(-1)^{\frac{1}{4}(p-1)}$ or $(2/p)$, for $\cos \frac{2n\pi}{p}$ is positive if $0 < n < \frac{1}{4}p$ and negative if $\frac{1}{4}p < n < \frac{1}{2}p$.

Therefore, since

$$(X_1)_{x=-1} \times (X_2)_{x=-1} = 1,$$

we have $(X_1)_{x=-1} = (X_2)_{x=-1} = (2/p)$.

6. For $p = 4n + 1$,

$$2 \sqrt{p} . x^{\frac{1}{2}} T = X_1 - X_2,$$

therefore $2 \sqrt{p} \left(\frac{1}{2} x^{-\frac{1}{2}} T + x^{\frac{1}{2}} \frac{dT}{dx} \right) = \frac{dX_1}{dx} - \frac{dX_2}{dx}$.

For $x = -1$, $T = 0$, therefore

$$2 \sqrt{p} . i \left(\frac{dT}{dx} \right)_{x=-1} = \left(\frac{dX_1}{dx} - \frac{dX_2}{dx} \right)_{x=-1}.$$

Now $\frac{dX_1}{dx} = \frac{X_1}{2} \left(\Sigma \frac{1}{x + x^{\frac{1}{2}} r^\alpha} + \Sigma \frac{1}{x - x^{\frac{1}{2}} r^\beta} \right)$,

therefore $\left(\frac{dX_1}{dx} \right)_{x=-1} = -\frac{1}{2} (2/p) \left(\Sigma \frac{1}{1 - i r^\alpha} + \Sigma \frac{1}{1 + i r^\beta} \right)$,

therefore

$$2 \sqrt{p} . i \left(\frac{dT}{dx} \right)_{x=-1} = -\tfrac{1}{2} (2/p) \left(\Sigma \frac{1}{1 - i r^\alpha} + \Sigma \frac{1}{1 + i r^\beta} - \Sigma \frac{1}{1 + i r^\alpha} - \Sigma \frac{1}{1 - i r^\beta} \right)$$
$$= -\tfrac{1}{2} (2/p) . 2i \sqrt{p} . h.$$

Therefore $$\left(\frac{dT}{dx}\right)_{x=-1} = -(2/p)\tfrac{1}{2}h.$$

By differentiating $\dfrac{x^p-1}{x-1} = S^2 - pxT^2$, and putting $x=-1$, we get

$$\left(\frac{dS}{dx}\right)_{x=-1} = -(2/p)\cdot\frac{p-1}{4}.$$

7. From $2\sqrt{p}.x^{\frac{1}{2}}T = X_1 - X_2$, we get

$$2\sqrt{p}.i\left(-\frac{dT}{dx}+\frac{d^2T}{dx^2}\right)_{x=-1} = \left(\frac{d^2X_1}{dx^2}-\frac{d^2X_2}{dx^2}\right)_{x=-1}.$$

Now

$$\frac{d^2X_1}{dx^2} = \frac{1}{2}\frac{dX_1}{dx}\left(\Sigma\frac{1}{x+x^{\frac{1}{2}}r^\alpha}+\Sigma\frac{1}{x-x^{\frac{1}{2}}r^\beta}\right)$$
$$-\frac{X_1}{2}\left\{\Sigma\frac{1+\frac{1}{2}x^{-\frac{1}{2}}r^\alpha}{(x+x^{\frac{1}{2}}r^\alpha)^2}+\Sigma\frac{1-\frac{1}{2}x^{-\frac{1}{2}}r^\beta}{(x-x^{\frac{1}{2}}r^\beta)^2}\right\}.$$

Substituting for $\dfrac{dX_1}{dx}$ in terms of X_1, and changing the signs inside the squared brackets,

$$\left(\frac{d^2X_1}{dx^2}\right)_{x=-1} = \left(\frac{X_1}{4}\right)_{x=-1}\left(\Sigma\frac{1}{1-ir^\alpha}+\Sigma\frac{1}{1+ir^\beta}\right)^2$$
$$-\left(\frac{X_1}{2}\right)_{x=-1}\left\{\Sigma\frac{1+\dfrac{r^\alpha}{2i}}{(1-ir^\alpha)^2}+\Sigma\frac{1-\dfrac{r^\beta}{2i}}{(1+ir^\beta)^2}\right\}.$$

Then, since
$$(X_1)_{x=-1} = (X_2)_{x=-1} = (2/p),$$
and
$$\left(\frac{dT}{dx}\right)_{x=-1} = -(2/p)\tfrac{1}{2}h,$$
we have

$$2i\sqrt{p}\left\{(2/p)\tfrac{1}{2}h+\left(\frac{d^2T}{dx^2}\right)_{x=-1}\right\}$$
$$=\tfrac{1}{4}(2/p)\left[\left(\Sigma\frac{1}{1-ir^\alpha}+\Sigma\frac{1}{1+ir^\beta}\right)^2-\left(\Sigma\frac{1}{1+ir^\alpha}+\Sigma\frac{1}{1-ir^\beta}\right)^2\right]$$
$$-\tfrac{1}{2}(2/p)\left\{\Sigma\frac{1+\dfrac{r^\alpha}{2i}}{(1-ir^\alpha)^2}+\Sigma\frac{1-\dfrac{r^\beta}{2i}}{(1+ir^\beta)^2}-\Sigma\frac{1-\dfrac{r^\alpha}{2i}}{(1+ir^\alpha)^2}-\Sigma\frac{1+\dfrac{r^\beta}{2i}}{(1-ir^\beta)^2}\right\}.$$

The coefficient of $\frac{1}{4}(2/p)$ is equal to $(p-1)\,2i\sqrt{p}\,.\,h$, and since

$$\Sigma\frac{1+\frac{r^{\alpha}}{2i}}{(1-ir^{\alpha})^2}-\Sigma\frac{1-\frac{r_{\alpha}}{2i}}{(1+ir^{\alpha})^2}=\Sigma\left\{\frac{1+\frac{r^{\alpha}}{2i}}{(1-ir^{\alpha})^2}-\frac{1-\frac{r^{-\alpha}}{2i}}{(1+ir^{-\alpha})^2}\right\}$$

$$=\Sigma\frac{i\,(r^{\alpha}+r^{-\alpha})}{(1-ir^{\alpha})\,(1+ir^{\alpha})}$$

$$=\Sigma\frac{1}{1-ir^{\alpha}}-\Sigma\frac{1}{1+ir^{\alpha}},$$

with a similar relation for the terms containing β, the coefficient of $-\frac{1}{2}(2/p)$ is equal to $2i\sqrt{p}\,.\,h$, and so

$$2i\sqrt{p}\left\{(2/p)\,\tfrac{1}{2}h+\left(\frac{d^2T}{dx^2}\right)_{x=-1}\right\}=(2/p)\,.\,2i\sqrt{p}\,.\,h\left(\frac{p-1}{4}-\frac{1}{2}\right),$$

or
$$\left(\frac{d^2T}{dx^2}\right)_{x=-1}=(2/p)\,.\,\frac{(p-5)\,h}{4}\,.$$

Proceeding as in the previous cases, we get

$$\left(\frac{d^2S}{dx^2}\right)_{x=-1}=(2/p)\left\{\frac{3\,(p-1)^2}{16}-\frac{ph^2}{4}\right\}.$$

Shrewsbury School.

ON VARIOUS EXPRESSIONS FOR h, THE NUMBER OF PROPERLY PRIMITIVE CLASSES FOR A DETERMINANT $-p$, WHERE p IS OF THE FORM $4n+3$, AND IS A PRIME OR THE PRODUCT OF DIFFERENT PRIMES.

(ADDITION TO THE SECOND PAPER.)

By *H. Holden.*

I. In a previous paper,* it is proved that, if q be prime to p,

$$\{q-(q/p)\}H=(q-1)\Sigma_0^{\frac{p}{q}}(a/p)+(q-2)\Sigma_{\frac{p}{q}}^{\frac{2p}{q}}(a/p)+\ldots+1\Sigma_{\frac{(q-2)p}{q}}^{\frac{(q-1)p}{q}}(a/p)$$

$$=(q-1)\Sigma_0^{\frac{p}{q}}(a/p)+(q-3)\Sigma_{\frac{p}{q}}^{\frac{2p}{q}}(a/p)+\ldots,$$

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* *Messenger of Mathematics*, Vol. xxxv., p. 102.

where $H = \dfrac{h}{2-(2/p)}$, and a is any positive integer.

It is now shown that similar relations hold when q is not prime to p, if in this case we put $(q/p)=0$.

II. Let $p = nP$ and $q = nQ$,

and let the integral parts of

$$\frac{P}{Q},\ \frac{2Q}{Q},\ \ldots,\ \frac{nQ.P}{Q}$$

be denoted by $\quad K_1,\ K_2,\ \ldots,\ K_q.$

Then since

$$r^{K_t} + r^{K_{Q+t}} + r^{K_{2Q+t}} + \ldots + r^{K_{(n-1)Q+t}}$$

$$= r^{K_t}\{1 + r^P + r^{2P} + \ldots + r^{(n-1)P}\}$$

$$= r^{K_t}.\frac{1-r^{nP}}{1-r^P} = 0 \text{ if } n \text{ is not equal to } 1,$$

it follows that

$$q\Sigma_1^{p-1}(a/p)\frac{1}{1-r^a} = \Sigma_1^{p-1}(a/p).\frac{1-r^{(K_1+1)a}}{1-r^a} + \Sigma_1^{p-1}(a/p).\frac{1-r^{(K_2+1)a}}{1-r^a} + \ldots$$

$$\ldots + \Sigma_1^{p-1}(a/p).\frac{1-r^{(K_q+1)a}}{1-r^a}.$$

The last sum being zero, we get

$$qH = \Sigma_0^{K_1}(a/p) + \Sigma_0^{K_2}(a/p) + \ldots + \Sigma_0^{K_{q-1}}(a/p)$$

$$= (q-1)\Sigma_0^{K_1}(a/p) + (q-2)\,\Sigma_{K_1+1}^{K_2}(a/p) + \ldots + 1\,\Sigma_{K_{q-2}+1}^{K_{q-1}}(a/p).$$

Generally, this reduces to

$$qH = (q-1)\,\Sigma_0^{K_1}(a/p) + (q-3)\,\Sigma_{K_1+1}^{K_2}(a/p) + \ldots,$$

the series terminating with the last positive coefficient.

If, however, $q \equiv 0 \pmod p$, we get

$$qH = (q=2)\Sigma_0^{K_1}(a/p) + (q-4)\Sigma_{K_1+1}^{K_2}(a/p) + \ldots.$$

III. Proceeding as in the previous paper, the following results are obtained for $p = 24 + 3$ (excepting $p = 3$), and $p = 24n + 15$.

(*a*) $$3H = 2\Sigma_0^{\frac{1}{3}p}(a/p).$$

(*b*) If a_r denotes the sum of the quadratic characters of

the integers between the integral parts of $(r-1)\frac{p}{12}+1$, and $\frac{rp}{12}$, we get the relations shown in the following table:

$p=$	$24n+3$	$24n+15$
$a_1+a_2=$	0	h
$a_3=$	0	0
$a_4=$	$\frac{1}{2}h$	$\frac{1}{2}h$
$a_5+a_6=$	$\frac{1}{2}h$	$-\frac{1}{2}h$
$a_1+a_5=$	$\frac{1}{2}h$	$\frac{1}{2}h$

(*c*) h is not greater than $\frac{p-3}{6}$, if $p=24n+3$ or $24n+15$.

(*d*) If a_r denotes the sum of the quadratic characters of the integers between the integral parts of $(r-1)\frac{p}{2n}+1$, and $\frac{rp}{2n}$, we have

$$a_1+a_3+a_5+\ldots=a_2+a_4+a_6+\ldots=\tfrac{1}{2}h,$$

the series being continued as far as the *nth* interval. These relations hold for all values of $p=4m+3$, not containing a square factor, when n is not prime to p.

Combining these results with those previously obtained, we get for all values of n,

$$a_1+a_3+a_5+\ldots=\tfrac{1}{2}h\{1+(n/p)\},$$
$$a_2+a_4+a_6+\ldots=\tfrac{1}{2}h\{1-(n/p)\},$$

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ON A THEOREM OF SEGAR'S.

By *Prof. E. J. Nanson.*

From the well-known formula of Jacobi for the quotient of a simple alternant by the corresponding difference product Professor Segar has, Vol. XXI., p. 148, deduced an extension to the effect that

$$|H_{\theta+\theta'}|=(-1)^{\frac{1}{2}n(n-1)}\Delta^{-2}a^{n-1}b^{n-1}\ldots(\alpha\beta\gamma\ldots)(\alpha'\beta'\gamma'\ldots),$$

where $(\alpha\beta\gamma\ldots)$ denotes the simple alternant whose n indices are $\alpha, \beta, \gamma, \ldots$ and n variables are $a, b, c, \ldots$; Δ is the differ-

ence product of $a, b, c, \ldots$, or, what is the same thing, is the value of $(\alpha\beta\gamma\ldots)$ when $\alpha, \beta, \gamma, \ldots$ have the values $n-1, n-2, n-3, \ldots$; H_r is the complete symmetric function of $a, b, c, \ldots$ of order r, and $|H_{\theta+\theta'}|$ denotes the determinant obtained by giving θ the values $\alpha, \beta, \gamma, \ldots$, and θ' the values $\alpha', \beta', \gamma', \ldots$. As Jacobi's theorem is generally proved by the method of generating functions the following elementary proof of Segar's theorem may be of interest, inasmuch as it also applies in the case of Jacobi's theorem.

If we multiply the columns of $(\alpha\beta\gamma\ldots)$ by the arbitrary multipliers $A, B, C, \ldots$ and multiply the result by $(\alpha'\beta'\gamma'\ldots)$ we have

$$ABC\ldots(\alpha\beta\gamma\ldots)(\alpha'\beta'\gamma'\ldots) = |K_{\theta\theta'}|,$$

where
$$K_{\theta\theta'} = Aa^{\theta+\theta'} + Bb^{\theta+\theta'} + \ldots.$$

Hence, if we take the multipliers $A, B, \ldots$, so that Aa^{1-n} is the co-factor of a^{n-1} in Δ, it is clear that $K_{\theta\theta'}$ is the value of $(\alpha\beta\gamma\ldots)$ when $\alpha = \theta + \theta' + n - 1$ and $\beta, \gamma, \ldots = n-2, n-3, \ldots$. Thus by a known theorem the value of $K_{\theta\theta'}$ is $\Delta H_{\theta+\theta'}$, provided $\theta+\theta'$ is positive. Clearly also when $\theta+\theta'$ is zero the value is Δ, and when $\theta+\theta'$ is negative but numerically less than n the value is zero. Hence, subject to the usual conventions that $H_0 = 1$ and that $H_r = 0$ when r is negative, we have

$$K_{\theta\theta'} = \Delta H_{\theta+\theta'},$$

provided that $\theta+\theta'$ is greater than $-n$.

Thus we have

$$ABC\ldots(\alpha\beta\gamma\ldots)(\alpha'\beta'\gamma'\ldots) = \Delta^n\,|H_{\theta+\theta'}|.$$

In this result, making

$$\alpha+\alpha' = s+s' = \ldots = 0,$$

and $\alpha', \beta', \gamma', \ldots, = 0, -1, -2, \ldots$, we find

$$ABC\ldots\Delta^2 a^{1-n} b^{1-n}\ldots(-1)^{\frac{1}{2}n(n-1)} = \Delta^n.$$

Hence, by division, we have

$$a^{n-1}b^{n-1}\ldots(\alpha\beta\gamma\ldots)(\alpha'\beta'\gamma'\ldots) = (-1)^{\frac{1}{2}n(n-1)}\Delta^2\,|H_{\theta+\theta'}|,$$

which is Segar's theorem.

If in this result we replace $\alpha', \beta', \gamma', \ldots$ by $0, -1, -2, \ldots$

$$(\alpha\beta\gamma\ldots) = (-1)^{\frac{1}{2}n(n-1)}\Delta\begin{vmatrix} H_\alpha, & H_\beta, & \ldots \\ H_{\alpha-1}, & H_{\beta-1}, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix},$$

and this is Jacobi's theorem.

ON THE NUMBER OF ABELIAN SUBGROUPS WHOSE ORDER IS A POWER OF A PRIME.

By *G. A. Miller*.

The present note aims to prove two fundamental facts with respect to the theorem that every group (G) whose order is divisible by p^m, p being a prime number, contains an abelian subgroup of order p^α whenever $m > \frac{1}{2}\alpha(\alpha - 1)$.* The first of these facts is that the number of the abelian subgroups of order p^α in G is always of the form $1 + kp$, and the second is that the given theorem can be extended since every group of order 64 contains an abelian subgroup of order 16. The proof of the latter will also serve to correct an error with respect to a published group which was supposed to be of order 64 and to contain no abelian subgroup of order 16.† We shall prove these two facts in the given order.

1. Let H be a Sylow subgroup of order p^m contained in G. Since H cannot transform a subgroup of order p^α into itself unless this subgroup is contained in H, it is only necessary to prove that the number of abelian subgroups of order p^α in H is of the form $1 + kp$. It will be convenient to note the following auxiliary theorems: A non-abelian group of order p^{s+1} contains 0, 1, or $p + 1$ abelian subgroups of order p^s; in the last case it contains p^{s-1} invariant operators. The number of subgroups of order p^s in any group of order p^{s-1} is of the form $1 + kp$. An abelian group of order p^s is contained either in none or in $1 + kp$ of the abelian subgroups of order p^{s+1} in H.

The first of these auxiliary theorems is practically self-evident, the second is a special case of a well-known extension of Sylow's theorem, while the third follows from the fact that all the operators of H which are commutative with all the operators of an abelian group of order p^s constitute a subgroup which includes all the abelian subgroups of order p^{s+1} in which the given group of order p^s occurs.

Let r_x represent the number of the invariant abelian subgroups of order p^{s+1} in which a given invariant abelian subgroup of order p^s occurs, while r_y represents the number of the invariant subgroups of order p^s in a given invariant abelian subgroup of order p^{s+1}. It was observed above that $r_x \equiv 1$,‡

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* *The Messenger of Mathematics*, Vol. XXVII. (1898), p. 120.

† *Bulletin of the American Mathematical Society*, Vol. III. (1896), p. 113.

‡ r_x cannot be 0, since $s < \alpha$. An invariant abelian subgroup of order $p^{\alpha-1}$ is contained in an invariant abelian subgroup of order p^α under H.

mod. p and $r_y \equiv 1$, mod. p. If we represent the total number of invariant abelian subgroups of order p^s and p^{s+1} in H by r_s and r_{s+1} respectively, we have the equation

$$\sum_{x=1}^{x=r_s} r_x = \sum_{y=1}^{y=r_{s+1}} r_y.$$

Each member of this equation represents the sum obtained by counting every invariant abelian subgroup of order p^{s+1} as many times as it contains invariant subgroups of order p^s. Hence $r_s \equiv r_{s+1}$, mod. p. This proves the theorem in question, since $r_1 \equiv 1$, mod. p. This result may be expressed as follows: *every group whoes order is divisible by p^m contains a multiple of p (which may be zero) non-abelian subgroups of order p^α whenever $m > \frac{1}{2}\alpha(\alpha-1)$.* In particular such a group contains an invariant abelian subgroup of order p^α.

2. To prove that every group of order 64 contains an abelian invariant subgroup of order 16 we may proceed as follows: Since every group of order 64 contains an invariant subgroup of order 4 it must also contain a subgroup (H) of order 32 which involves at least 4 invariant operators. These are included in an abelian subgroup (K) of order 8 which is invariant under H. If H does not include an abelian group of order 16 these four invariant operators constitute the four-group. The square of every operator of H is in this four-group, since this square is commutative with every operator of K. The quotient group of H with respect to this four-group is therefore the abelian group of order 8 and of type (1, 1, 1).

Any subgroup (H_1) of order 16 which includes K has a commutator subgroup whose order cannot exceed 2. Let t be any operator of H which is not found in H_1. As we may assume that t is non-commutative with twelve of the operators of H_1, it must transform some of the operators of H_1 into themselves multiplied by the commutator of order 2 in H_1 in case H_1 is non-abelian. As such operators have only two conjugates under H they are invariant under 16 operators of H. Hence H contains a subgroup of order 16 which involves at least 8 invariant operators. Such a subgroup is clearly abelian. That is, *every group of order 64 contains an abelian subgroup of order 16.* If there is more than one such subgroup there is an odd number according to the above proof. Hence at least one is invariant.

A NOTE ON STIRLING'S SERIES AND EULER'S CONSTANT.

By *T. J. I'a. Bromwich*, Queen's College, Galway.

THE following investigation, although elementary, enables us to find with considerable ease the first few terms in Stirling's series for $\log(n!)$; and it seems to be shorter than any other proof with which I am acquainted. The idea of summing up the left-hand sides of equations (2) was suggested by Chrystal's method (*Algebra*, vol. II, ch. 30, § 17), but the details of calculation are entirely different.

Consider first the integral

$$\int_0^b \frac{2t^2}{a^2-t^2}\,dt = \int_0^b \left(\frac{2a^2}{a^2-t^2} - 2\right) dt = a\log\frac{a+b}{a-b} - 2b.$$

Thus we have

$$(1) \qquad \int_0^{\frac{1}{2}} \frac{2t^2}{(r-\frac{1}{2})^2 - t^2}\,dt = (r-\tfrac{1}{2})\log\left(\frac{r}{r-1}\right) - 1.$$

Write in equation (1), $r = n,\ n-1,\ n-2,\ \dots,\ 2$, and then we find

$$(2) \quad \left\{ \begin{aligned} &(n-\tfrac{1}{2})\log\left(\frac{n}{n-1}\right) = 1 + \int_0^{\frac{1}{2}} \frac{2t^2}{(n-\frac{1}{2})^2 - t^2}\,dt, \\ &(n-\tfrac{3}{2})\log\left(\frac{n-1}{n}\right) = 1 + \int_0^{\frac{1}{2}} \frac{2t^2}{(n-\frac{3}{2})^2 - t^2}\,dt, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ &\tfrac{5}{2}\log(\tfrac{3}{2}) \qquad = 1 + \int_0^{\frac{1}{2}} \frac{2t^2}{(\frac{5}{2})^2 - t^2}\,dt, \\ &\tfrac{3}{2}\log(\tfrac{2}{1}) \qquad = 1 + \int_0^{\frac{1}{2}} \frac{2t^2}{(\frac{3}{2})^2 - t^2}\,dt. \end{aligned} \right.$$

On adding up equations (2) we find

$$(n-\tfrac{1}{2})\log n - [\log(n-1) + \log(n-2) + \dots + \log 3 + \log 2]$$
$$= (n-1) + \int_0^{\frac{1}{2}} \left(\sum_{r=2}^{n} \frac{2t^2}{(r-\frac{1}{2})^2 - t^2}\right) dt,$$

or

$$(3) \qquad (n+\tfrac{1}{2})\log n - \log(n!) = (n-1) + \int_0^{\frac{1}{2}} \left(\sum_{r=2}^{n} \frac{2t^2}{(r-\frac{1}{2})^2 - t^2}\right) dt.$$

Now the function of t

$$f(t)=\sum_2^\infty \frac{2t^2}{(r-\frac{1}{2})^2-t^2}$$

is obviously continuous in the interval $(0, \frac{1}{2})$, because the series converges uniformly in that interval, the terms of the series being less than those of the convergent series of postive terms $\sum_2^\infty 1/(r-1)^2$.

Thus $\int_0^{\frac{1}{2}} f(t)\,dt$ is a finite constant K, say.

But $$\sum_2^n \frac{2t^2}{(r-\frac{1}{2})^2-t^2}=f(t)-\sum_{n+1}^\infty \frac{2t^2}{(r-\frac{1}{2})^2-t^2},$$

and $$\frac{2t^2}{(r-\frac{1}{2})^2-t^2}\leqq\frac{2t^2}{r(r-1)}, \text{ because } t^2\leqq\tfrac{1}{4},$$

so that $$\sum_2^n \frac{2t^2}{(r-\frac{1}{2})^2-t^2}\geqq f(t)-\sum_{n+1}^\infty \frac{2t^2}{r(r-1)},$$

or $$\geqq f(t)-\frac{2t^2}{n}.$$

Hence

$$\int_0^{\frac{1}{2}}\left(\sum_2^n \frac{2t^2}{(r-\frac{1}{2})^2-t^2}\right)dt\geqq K-\frac{1}{12n}. \tag{4}$$

Again

$$\frac{2t^2}{r(r-1)}-\frac{2t^2}{(r-\frac{1}{2})^2-t^2}=\frac{2t^2(\frac{1}{4}-t^2)}{r(r-1)[(r-\frac{1}{2})^2-t^2]}\leqq\frac{2t^2(\frac{1}{4}-t^2)}{r^2(r-1)^2},$$

so that $$\sum_{n+1}^\infty\left[\frac{2t^2}{r(r-1)}-\frac{2t^2}{(r-\frac{1}{2})^2-t^2}\right]\leqq\sum_{n+1}^\infty\frac{2t^2(\frac{1}{4}-t^2)}{r^2(r-1)^2},$$

or $$\sum_2^n \frac{2t^2}{(r-\frac{1}{2})^2-t^2}\leqq f(t)-\frac{2t^2}{n}+\sum_{n+1}^\infty\frac{2t^2(\frac{1}{4}-t^2)}{r^2(r-1)^2}.$$

Now $$\sum_{n+1}^\infty\frac{1}{r^2(r-1)^2}<\int_{n+1}^\infty\frac{dx}{(x-1)^4},$$

or $$\sum_{n+1}^\infty\frac{1}{r^2(r-1)^2}<\frac{1}{3n^3},$$

Thus $$\sum_2^n \frac{2t^2}{(r-\frac{1}{2})^2-t^2}\leqq f(t)-\frac{2t^2}{n}+\frac{2t^2(\frac{1}{4}-t^2)}{3n^3},$$

where, of course, it is to be remembered that t is restricted to the interval $(0, \frac{1}{2})$. It is now evident that

$$(5) \qquad \int_0^{\frac{1}{2}} \left[\sum_2^n \frac{2t^2}{(r-\frac{1}{2})^2 - t^2} \right] dt \leqq K - \frac{1}{12n} + \frac{1}{360n^3},$$

because $$\int_0^{\frac{1}{2}} \tfrac{2}{3}t^2 (\tfrac{1}{4} - t^2)\, dt = \left[\tfrac{1}{18}t^3 - \tfrac{2}{15}t^5 \right]_0^{\frac{1}{2}} = \tfrac{1}{360}.$$

From (3), (4), and (5) we see that

$$(6) \quad (n+\tfrac{1}{2}) \log n - \log (n!) = n + K - 1 - \frac{1}{12n} + \frac{\theta}{360n^3},$$

where θ is some real number between 0 and 1.

The value of $(K-1)$ is found to be $-\frac{1}{2}\log(2\pi)$ by using Wallis's Theorem in the manner given by Chrystal; but it can also be found independently by observing that as a matter of fact

$$f(t) = \pi t \tan (\pi t) - \frac{2t^2}{\frac{1}{4} - t^2} = \pi t \tan (\pi t) + 2 - \frac{2}{1 - 4t^2},$$

which gives on integration

$$(7) \qquad K = \textstyle\int_0^{\frac{1}{2}} f(t)\, dt = 1 - \frac{1}{2} \log (2\pi).$$

EULER'S CONSTANT.

A similar method can be applied without much difficulty to obtain an approximation to Euler's constant.

We find, by calculating the integral, that

$$\int_0^b \frac{4at^2}{(a^2 - t^2)^2}\, dt = b \left(\frac{1}{a-b} + \frac{1}{a+b} \right) - \log \left(\frac{a+b}{a-b} \right).$$

Thus, on taking $a = r - \frac{1}{2}$, $b = \frac{1}{2}$, we find

$$(8) \qquad \int_0^{\frac{1}{2}} \frac{4t^2 (r - \frac{1}{2})\, dt}{[(r-\frac{1}{2})^2 - t^2]^2} = \tfrac{1}{2} \left(\frac{1}{r-1} + \frac{1}{r} \right) - \log \left(\frac{r}{r-1} \right).$$

Write in equation (8), $r = n, n-1, \ldots, 2$, and add; then we have

$$(9) \qquad \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right) - \log n$$

$$= \tfrac{1}{2} \left(1 + \frac{1}{n} \right) + \int_0^{\frac{1}{2}} \left[\sum_{r=2}^{n} \frac{4t^2 (r-\frac{1}{2})}{\{(r-\frac{1}{2})^2 - t^2\}^2} \right] dt$$

$$= C + \frac{1}{2n} - \int_0^{\frac{1}{2}} \left[\sum_{n+1}^{\infty} \frac{4t^2 (r-\frac{1}{2})}{\{(r-\frac{1}{2})^2 - t^2\}^2} \right] dt,$$

where the constant C can be proved to be finite by an argument exactly similar to the one used for K in the previous work.

Write, for brevity, $R = r(r-1)$, $T = \frac{1}{4} - t^2$, so that R, T are both positive in the integral. Thus, in the integral

$$(r-\tfrac{1}{2})^2 - t^2 = R + T,$$

$$\frac{1}{(R+T)^2} \leqq \frac{1}{R^2},$$

and
$$\frac{1}{R^2} - \frac{1}{(R+T)^2} = \frac{2T(R+\frac{1}{2}T)}{R^2(R+T)^2} \leqq \frac{2T}{R^3}.$$

Hence we have

$$(10) \qquad \int_0^{\frac{1}{2}} \left[\sum_{n+1}^{\infty} \frac{4t^2(r-\frac{1}{2})}{(R+T)^2}\right] dt \leqq \int_0^{\frac{1}{2}} \left[\sum_{n+1}^{\infty} \frac{4t^2(r-\frac{1}{2})}{R^2}\right] dt,$$

and

$$(11) \qquad \geqq \int_0^{\frac{1}{2}} \left[\sum_{n+1}^{\infty} \frac{4t^2(r-\frac{1}{2})}{R^2}\right] dt - \int_0^{\frac{1}{2}} \left[\sum_{n+1}^{\infty} \frac{8t^2T(r-\frac{1}{2})}{R^3}\right] dt.$$

But $$\sum_{n+1}^{\infty} \frac{2r-1}{R^2} = \sum_{n+1}^{\infty} \frac{r^2-(r-1)^2}{r^2(r-1)^2} = \sum_{n+1}^{\infty} \left[\frac{1}{(r-1)^2} - \frac{1}{r^2}\right] = \frac{1}{n^2},$$

and
$$\int_0^{\frac{1}{2}} 2t^2 dt = \tfrac{1}{12},$$

so that
$$\int_0^{\frac{1}{2}} \left[\sum_{n+1}^{\infty} \frac{4t^2(r-\frac{1}{2})}{R^2}\right] dt = \frac{1}{12n^2},$$

Also
$$\sum_{n+1}^{\infty} \frac{(r-\frac{1}{2})}{R^3} < \int_{n+1}^{\infty} \frac{dx}{(x-1)^5} = \frac{1}{4n^4},$$

and
$$\int_0^{\frac{1}{2}} 8t^2 T dt = \int_0^{\frac{1}{2}} 8t^2(\tfrac{1}{4} - t^2)\, dt = \tfrac{1}{30},$$

so that
$$\int_0^{\frac{1}{2}} \left[\sum_{n+1}^{\infty} \frac{8t^2T(r-\frac{1}{2})}{R^3}\right] dt < \frac{1}{120n^4}.$$

Combining these results with (9), (10), and (11), we find that

$$(12) \qquad \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \log n = C + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\theta'}{120n^4},$$

where θ' is a real number between 0 and 1.

From (12) it is evident that

$$C = \lim_{n=\infty} \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \log n \right],$$

or that C is Euler's constant.

By taking $n = 10$, say, we can get a very fair approximation to C from equation (12); thus we have

$$(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{10}) - \log 10 = \cdot 62638316,$$

$$\frac{1}{2n} = \cdot 05, \quad \frac{1}{12n^2} = \cdot 00083333, \quad \frac{1}{120n^4} = \cdot 00000083.$$

These results give

$$\cdot 5771256 < C < \cdot 5771265,$$

and the approximation could easily be improved by taking larger values for n.

Of course both equations (6) and (12) are well known,* but they do not seem to have been found previously except by the aid of more elaborate analysis.

THE PERSYMMETRIC DETERMINANT WHOSE ELEMENTS ARE IN HARMONICAL PROGRESSION.

By *Thomas Muir, LL.D.*

1. In a paper of Murphy's† there appears a set of linear equations whose determinant is a special case of

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a+d} & \frac{1}{a+2d} & \cdots & \frac{1}{a+(n-1)d} \\ \frac{1}{a+d} & \frac{1}{a+2d} & \frac{1}{a+3d} & \cdots & \frac{1}{a+nd} \\ \frac{1}{a+2d} & \frac{1}{a+3d} & \frac{1}{a+4d} & \cdots & \frac{1}{a+(n+1)d} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{a+(n-1)d} & \cdots & \cdots & \cdots & \frac{1}{a+2(n-1)d} \end{vmatrix},$$

* See, for instance, Seliwanoff, *Differenzenrechnung*, §§ 12, 43.

† Murphy, R. "On elimination between an indefinite number of unknown quantities." *Trans. Cambridge Phil. Soc.*, v., pp. 65–76.

or say, for shortness sake,

$$P\left(\frac{1}{a},\ \frac{1}{a+d},\ \frac{1}{a+2d},\ \ldots,\ \frac{1}{a+2(n-1)d}\right),\quad \text{or}\quad P_n.$$

Since, in the case of any determinant D we have the identity

$$D = \begin{vmatrix} D_1^1 & D_1^n \\ D_n^1 & D_n^n \end{vmatrix} \div D_{1,n}^{1,n},$$

where D_r^s means the determinant obtained from D by deleting the r^{th} row and s^{th} column; and, since in the particular case of P_n the five minors appearing on the right of the identity are of the same form as the originating determinant on the left, it is clear that the evaluation of P_n can be accomplished by so-called 'mathematical induction.' The result obtained is

$$\frac{\{(n-1)!\,d^{n-1}.(n-2)!\,d^{n-2}\ldots 2!\,d^2.1!\,d^1\}^2}{a\,(a+d)^2(a+2d)^3\ldots\{a+(n-1)d\}^n(a+nd)^{n-1}\ldots\{a+(2n-2)d\}} \quad\text{......(I.)},$$

where the denominator is the product of the n^2 denominators of the elements of the determinant and may therefore be put in the alternative form

$$\Pi\,(a,\,d)\,.\,\Pi\,(a+d,\,d)\,.\,\Pi\,(a+2d,\,d)\ldots\Pi\,\{a+(n-1)\,d,\,d\}$$

if we use $\Pi\,(a,d)$ to stand for the product of the denominators of the first row or first column.

2. Perhaps the most instructive mode of proceeding is to bring out the latent connection which exists between the given determinant and the determinant

$$\begin{vmatrix} 1 & a & a\,(a+d) & a\,(a+d)\,(a+2d) & \ldots \\ 1 & a+d & (a+d)\,(a+2d) & (a+d)\,(a+2d)\,(a+3d) & \ldots \\ 1 & a+2d & (a+2d)\,(a+3d) & (a+2d)\,(a+3d)\,(a+4d) & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix},$$

or Q_n say. The latter, if we perform on it the operations

$$\text{row}_n - \text{row}_{n-1},\quad \text{row}_{n-1} - \text{row}_{n-2},\quad \ldots$$

is changeable into a determinant of the next lower order, having d, $2d$, $3d$, ... as factors of its 1st, 2nd, 3rd, ... columns respectively, and such that when those factors are removed the resulting determinant is perfectly similar in form to Q_n.

We thus obtain

$$Q_n = d^{n-1}(n-1)!\,.\,Q_{n-1},$$

and thence ultimately

$$Q_n = d^{n-1}(n-1)!\,.\,d^{n-2}(n-2)!\ldots d^2\,2!\,.\,d\,1! \quad \ldots\ldots\text{(II.)},$$

an expression, be it remarked, which is independent of a.

3. Returning now to P_n and noting that from the theory of Finite Differences

$$\frac{1}{a} - \frac{C_{s,1}}{a+d} + \frac{C_{s,2}}{a+2d} - \ldots + (-)^{s-1}\frac{C_{s,s-1}}{a+(s-1)d} + (-)^s\frac{C_{s,s}}{a+sd}$$

$$= \frac{d^s s!}{a(a+d)(a+2d)\ldots(a+sd)},$$

and that the changing of a into $a+d$ affects the denominators only, we perform the operations

$$\begin{aligned} \mathrm{col}_1 - C_{n-1,1}\mathrm{col}_2 + C_{n-1,2}\mathrm{col}_3 - C_{n-1,3}\mathrm{col}_4 + \ldots&, \\ \mathrm{col}_2 - C_{n-2,1}\mathrm{col}_3 + C_{n-2,2}\mathrm{col}_4 - \ldots&, \\ \mathrm{col}_3 - C_{n-3,1}\mathrm{col}_4 + \ldots&, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots&, \end{aligned}$$

and thus obtain a determinant having $d^{n-1}(n-1)!$ as a factor of the first column, $d^{n-2}(n-2)!$ as a factor of the second column, and so on. If we remove those factors, and thereafter remove from the rows the factors

$$1/\Pi(a, d),\ \ 1/\Pi(a+d, d),\ \ \ldots,$$

the resulting determinant is Q_n, and we thus arrive at

$$P_n = \frac{d^{n-1}(n-1)!\,.\,d^{n-2}(n-2)!\ldots d^2\,2!\,.\,d^1\,1!}{\Pi(a, d)\,.\,\Pi(a+d, d)\ldots\Pi\{a+(n-1)d, d\}}\,.\,Q_n$$

$$= \frac{\{d^{n-1}(n-1)!\,.\,d^{n-2}(n-2)!\ldots d^2\,2!\,.\,d\,1!\}^2}{\Pi(a, d)\,.\,\Pi(a+d, d)\ldots\Pi\{a+(n-1)d, d\}}$$

as desired.

4. Taking as our illustration the case where $n=4$ and performing the operations

$$\mathrm{col}_1 - 3\,\mathrm{col}_2 + 3\,\mathrm{col}_3 - \mathrm{col}_4,\quad \mathrm{col}_2 - 2\,\mathrm{col}_3 + \mathrm{col}_4,\quad \mathrm{col}_3 - \mathrm{col}_4,$$

we obtain

$$P_4 = \begin{vmatrix} \dfrac{d^3.3!}{a(a+d)(a+2d)(a+3d)} & \dfrac{d^2.2!}{(a+d)(a+2d)(a+3d)} & \dfrac{d.1!}{(a+2d)(a+3d)} & \dfrac{1}{a+6d} \\ \dfrac{d^3.3!}{(a+d)(a+2d)(a+3d)(a+4d)} & \dfrac{d^2.2!}{(a+2d)(a+3d)(a+4d)} & \dfrac{d.1!}{(a+3d)(a+4d)} & \dfrac{1}{a+5d} \\ \dfrac{d^3.3!}{(a+2d)(a+3d)(a+4d)(a+5d)} & \dfrac{d^2.2!}{(a+3d)(a+4d)(a+5d)} & \dfrac{d.1!}{(a+4d)(a+5d)} & \dfrac{1}{a+5d} \\ \dfrac{d^3.3!}{(a+3d)(a+4d)(a+5d)(a+6d)} & \dfrac{d^2.2!}{(a+4d)(a+5d)(a+6d)} & \dfrac{d.1!}{(a+5d)(a+6d)} & \dfrac{1}{a+6d} \end{vmatrix}$$

$$= \frac{d^3 3!.d^2 2!.d\,1!}{a(a+d)^2(a+2d)^3(a+3d)^4(a+4d)^3(a+5d)^2(a+6d)} \times$$

$$\begin{vmatrix} 1 & a & a(a+d) & a(a+d)(a+2d) \\ 1 & a+d & (a+d)(a+2d) & (a+d)(a+2d)(a+3d) \\ 1 & a+2d & (a+2d)(a+3d) & (a+2d)(a+3d)(a+4d) \\ 1 & a+3d & (a+3d)(a+4d) & (a+3d)(a+4d)(a+5d) \end{vmatrix}.$$

Then, reducing the last determinant by means of the operations

$$\text{row}_4 - \text{row}_3, \quad \text{row}_3 - \text{row}_2, \quad \text{row}_2 - \text{row}_1,$$

we find its equivalent to be

$$\begin{vmatrix} d & 2d(a+d) & 3d(a+d)(a+2d) \\ d & 2d(a+2d) & 3d(a+2d)(a+3d) \\ d & 2d(a+3d) & 3d(a+3d)(a+4d) \end{vmatrix},$$

and thence to be

$$d^3 3!.d^2 2!.d\,1!,$$

so that finally there results

$$P_4 = \frac{\{d^3 3!.d^2 2!.d\,1!\}^2}{a(a+d)^2 \ldots (a+5d)^2 (a+6d)}.$$

5. Since both the set of column-operations and the set of row-operations in the preceding can be condensed into multiplication by a determinant, we may represent* the whole process by

$$P_4 \times_{rr} \begin{vmatrix} 1 & -3 & 3 & -1 \\ . & 1 & -2 & 1 \\ . & . & 1 & -1 \\ . & . & . & 1 \end{vmatrix} \times \frac{a(a+d)^2 \ldots (a+5d)^2 (a+6d)}{d^3.3!.d^2.2!.d.1!} \times_{cc} \begin{vmatrix} 1 & -1 & 1 & -1 \\ . & 1 & -2 & 3 \\ . & . & 1 & -3 \\ . & . & . & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & . & . & . \\ a & d & . & . \\ a(a+d) & 2d(a+d) & d^2.2! & . \\ a(a+d)(a+2d) & 3d(a+d)(a+2d) & 6d^2(a+2d) & d^3.3! \end{vmatrix} \quad \ldots\ldots\ldots\ldots(\text{III.}),$$

* Multiplication in row-by-row fashion is here denoted by $\times_{rr}$, the others being $\times_{rc}$, $\times_{cr}$, $\times_{cc}$. The proper symbol for matrix-multiplication is thus $\times_{rc}$.

which is seen at a glance to involve the result

$$P_4 = \frac{\{d^3 3!.d^2 2!.d 1!\}^2}{a(a+d)^2 \dots (a+5d)^2 (a+6d)} .$$

An alternative process is

$$\begin{vmatrix} 1 & . & . & . \\ 1 & 1 & . & . \\ 1 & 2 & 1 & . \\ 1 & 3 & 3 & 1 \end{vmatrix} \times_{rr} \begin{vmatrix} 1 & . & . & . \\ a & d & . & . \\ a(a+d) & 2d(a+d) & d^2.2! & . \\ a(a+d)(a+2d) & 3d(a+d)(a+2d) & 6d^2(a+2d) & d^3.3! \end{vmatrix},$$

$$\times_{rr} \begin{vmatrix} d^3.3! & 6d^2 & 3d & 1 \\ . & d^2.2! & 2d & 1 \\ . & . & d.1! & 1 \\ . & . & . & 1 \end{vmatrix} = P_4 \times \Pi(a, d).\Pi(a+d, d) \dots \Pi(a+3d, d) \text{.........(IV.)}$$

6. If to the square array of elements in Q_n we annex an $(n+1)^{th}$ column, following the same law of formation as in the case of the other columns, we obtain an n-by-$(n+1)$ array; and if from this we delete the $n+1$ columns in succession there results a series of $n+1$ square arrays. Calling the non-quadrate array A, and denoting by A_r the determinant of the array which results from A on deleting the r^{th} column,

we find by following the process of § 2 in each individual case that

$$A_{n+1} = Q_n,$$
$$A_n = Q_n . \{a + (n-1)d\}\ C_{n,1},$$
$$A_{n-1} = Q_n . \{a + (n-1)d\} . \{a + (n-2)d\}\ C_{n,2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$A_1 = Q_n . \{a + (n-1)d\} \ldots (a+d)\, a \ldots\ldots\ldots\ldots (\text{V}).$$

A satisfactory proof of this, that is to say, a proof dealing once for all with A_{n-r}, is desirable.

From the fact that the next Q is expressible in terms of the A's and the elements of its $(n+1)^{\text{th}}$ row, namely,

$$\left.\begin{aligned} Q_{n+1} = \{a+nd\}\ \{a+(n+1)d\} \ldots \{a+(2n-1)d\} . A_{n+1} \\ - \{a+nd\} \ldots \{a+(2n-2)d\} . A_n \\ + \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ + (-)^n . A_1 \end{aligned}\right\},$$

another identity in Finite Differences makes its appearance. Thus, when $n = 3$, we have

$$Q_4 = (a+3d)\ (a+4d)\ (a+5d) A_4$$
$$- (a+3d)\ (a+4d)\ A_3 + (a+3d)\ A_2 - A_1,$$

and therefore, on dividing by Q_3,

$$d^3 . 3! = (a+3d)\ (a+4d)\ (a+5d) - 3\ (a+2d)\ (a+3d)\ (a+4d)$$
$$+ 3\ (a+d)\ (a+2d)\ (a+3d) - a\ (a+d)\ (a+2d).$$

7. By dealing with the rows of Q_n as we dealt with the columns in § 6 we obtain an $(n+1)$-by-n array and another set of $n+1$ determinants. Calling this array A', we find the values of these to be

$$A'_{n+1} = Q_n,$$
$$A'_n = Q_n . C_{n,1},$$
$$A'_{n-1} = Q_n . C_{n,2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots,$$
$$A_1' = Q_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{VI.}).$$

That $A'_{n+1} = A_1'$ is evident from the fact that Q_n is independent of a. Similarly, having proved that A_n' is equal to

$Q_n . C_{n,1}$, we can show that A_3' is the same. Thus, by definition

$$A_3' = \begin{vmatrix} 1 & a & a(a+d) \\ 1 & a+d & (a+d)(a+2d) \\ 1 & a+3d & (a+3d)(a+4d) \end{vmatrix},$$

and, since its equivalent $Q_3 . C_{3,1}$ or $d^2 2!.d1!.3$ is independent of a, the change of a into $a+3d$ and d into $-d$ gives us

$$-d^2 2!.d1!.3 = \begin{vmatrix} 1 & a+3d & (a+3d)(a+2d) \\ 1 & a+2d & (a+2d)(a+d) \\ 1 & a & a(a-d) \end{vmatrix},$$

which, when the third column is increased by $2d$ times the second, becomes

$$\begin{vmatrix} 1 & a+3d & (a+3d)(a+4d) \\ 1 & a+2d & (a+2d)(a+3d) \\ 1 & a & a(a+d) \end{vmatrix}.$$

As this by definition is $-A_3'$ it follows that $A_3' = A_3'$.

8. If now we treat P_n as Q_n has been treated in § 6, denoting the n-by-$(n+1)$ array so obtained by B, there is found by following the process of § 3

$$B_{n+1} = \frac{d^{n-1}(n-1)!.d^{n-2}(n-2)!\ldots d^2 2!.d1!}{\Pi_{n+1}} . A'_{n+1},$$

$$B_n = \frac{d^{n-1}(n-1)!.d^{n-2}(n-2)!\ldots d^2 2!.d1!}{\Pi_n} . A'_n,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where Π_r denotes the product of the denominators of all the elements of B_r. Inserting the values of the A''s, and putting Q_n for the common numerator, we have

$$B_{n+1} = \frac{(Q_n)^2}{\Pi_{n+1}},$$

$$B_n = \frac{(Q_n)^2}{\Pi_n} . C_{n,1},$$

$$B_{n-1} = \frac{(Q_n)^2}{\Pi_{n-1}} . C_{n,2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$B_1 = \frac{(Q_n)^2}{\Pi_1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(VII.)}.$$

It only remains to be noted that the $(n+1)$-by-n array, which by analogy we should call B', gives rise to the same series of determinants as the array B.

Capetown, S.A.,
9th May, 1906.

THE JACOBIAN OF THE PRIMARY MINORS OF A CIRCULANT.

By *Thomas Muir, LL.D.*

1. If a be one of the elements of a circulant of the n^{th} order, and A the signed primary minor corresponding to a in one of the positions which a occupies, then A is the signed primary minor corresponding to a in any other one of its positions. The number of different primary minors of a circulant of the n^{th} order is thus n; the adjugate of $C(a_1, a_2, \ldots, a_n)$ is $C(A_1, A_2, \ldots, A_n)$: and the Jacobian of the signed primary minors is

$$\frac{\partial(A_1, A_2, \ldots, A_n)}{\partial(a_1, a_2, \ldots, a_n)}.$$

2. Taking, for shortness' sake, the circulant of the fifth order, $C(a, b, c, d, e)$, we have

$$A = \begin{vmatrix} a & b & c & d \\ e & a & b & c \\ d & e & a & b \\ c & d & e & a \end{vmatrix},$$

and

$$\frac{\partial A}{\partial a} = \begin{vmatrix} 1 & b & c & d \\ . & a & b & c \\ . & e & a & b \\ . & d & e & a \end{vmatrix} + \begin{vmatrix} a & . & c & d \\ e & 1 & b & c \\ d & . & a & b \\ c & . & e & a \end{vmatrix} + \begin{vmatrix} a & b & . & d \\ e & a & . & c \\ d & e & 1 & b \\ c & d & . & a \end{vmatrix} + \begin{vmatrix} a & b & c & . \\ e & a & b & . \\ d & e & a & . \\ c & d & e & 1 \end{vmatrix},$$

$$\frac{\partial A}{\partial b}=\begin{vmatrix} \cdot & b & c & d \\ \cdot & a & b & c \\ \cdot & e & a & b \\ \cdot & d & e & a \end{vmatrix}+\begin{vmatrix} a & 1 & c & a \\ e & \cdot & b & c \\ d & \cdot & a & b \\ c & \cdot & e & a \end{vmatrix}+\begin{vmatrix} a & b & \cdot & d \\ e & a & 1 & c \\ d & e & \cdot & b \\ c & d & \cdot & a \end{vmatrix}+\begin{vmatrix} a & b & c & \cdot \\ e & a & b & \cdot \\ d & e & a & 1 \\ c & d & e & \cdot \end{vmatrix},$$

$$\frac{\partial A}{\partial c}=\begin{vmatrix} \cdot & b & c & d \\ \cdot & a & b & c \\ \cdot & e & a & b \\ 1 & d & e & a \end{vmatrix}+\begin{vmatrix} a & \cdot & c & d \\ e & \cdot & b & c \\ d & \cdot & a & b \\ c & \cdot & e & a \end{vmatrix}+\begin{vmatrix} a & b & 1 & d \\ e & a & \cdot & c \\ d & e & \cdot & b \\ c & d & \cdot & a \end{vmatrix}+\begin{vmatrix} a & b & c & \cdot \\ e & a & b & 1 \\ d & e & a & \cdot \\ c & d & e & \cdot \end{vmatrix},$$

$$\frac{\partial A}{\partial d}=\begin{vmatrix} \cdot & b & c & d \\ \cdot & a & b & c \\ 1 & e & a & b \\ \cdot & d & e & a \end{vmatrix}+\begin{vmatrix} a & \cdot & c & d \\ e & \cdot & b & c \\ d & \cdot & a & b \\ c & 1 & e & a \end{vmatrix}+\begin{vmatrix} a & b & \cdot & d \\ e & a & \cdot & c \\ d & e & \cdot & b \\ c & d & \cdot & a \end{vmatrix}+\begin{vmatrix} a & b & c & 1 \\ e & a & b & \cdot \\ d & e & a & \cdot \\ c & d & e & \cdot \end{vmatrix},$$

$$\frac{\partial A}{\partial e}=\begin{vmatrix} \cdot & b & c & d \\ 1 & a & b & c \\ \cdot & e & a & b \\ \cdot & d & e & a \end{vmatrix}+\begin{vmatrix} a & \cdot & c & d \\ e & \cdot & b & c \\ d & 1 & a & b \\ c & \cdot & e & a \end{vmatrix}+\begin{vmatrix} a & b & \cdot & d \\ e & a & \cdot & c \\ d & e & \cdot & b \\ c & d & 1 & a \end{vmatrix}+\begin{vmatrix} a & b & c & \cdot \\ e & a & b & \cdot \\ d & e & a & \cdot \\ c & d & e & \cdot \end{vmatrix},$$

where, necessarily, the determinants occupying the first places in the right-hand members differ only in their first columns, those occupying the second places differ only in their second columns, and so on. If now, after noting that the multiplication of any right-hand member is effected by merely substituting the multiplier for the element 1, we use in connection with the five equations the multipliers a, b, c, d, e, and sum the results, we obtain

$$a\frac{\partial A}{\partial a}+b\frac{\partial A}{\partial b}+\ldots+e\frac{\partial A}{\partial e}=(n-1)A,$$

as, of course, we ought to expect. But if the multipliers be b, c, d, e, a, the sum on the right is

$$\begin{vmatrix} b & b & c & d \\ a & a & b & c \\ e & e & a & b \\ d & d & e & a \end{vmatrix}+\begin{vmatrix} a & c & c & d \\ e & b & b & c \\ d & a & a & b \\ c & c & e & a \end{vmatrix}+\begin{vmatrix} a & b & d & d \\ e & a & c & c \\ d & e & b & b \\ c & d & e & e \end{vmatrix}+\begin{vmatrix} a & b & c & e \\ e & a & b & d \\ d & e & a & c \\ c & d & e & b \end{vmatrix},$$

i.e. $\quad 0 \quad + \quad 0 \quad + \quad 0 \quad - \quad E \quad ;$

if the multipliers be c, d, e, a, b the sum is

$$0 \quad + \quad 0 \quad - \quad D \quad + \quad 0 \quad ;$$

if the multipliers be d, e, a, b, c the sum is

$$0 \quad - \quad C \quad + \quad 0 \quad + \quad 0 \quad ;$$

and if the multipliers be e, a, b, c, d the sum is

$$-B \quad + \quad 0 \quad + \quad 0 \quad + \quad 0 \quad .$$

Using $\overset{\circ}{\Sigma}$ to denote the sum of the terms obtained by performing the cyclic change of a into b, b into c, ..., e into a, we may write these results in the form

$$\overset{\circ}{\Sigma} a \frac{\partial A}{\partial a} = (n-1) A,$$

$$\overset{\circ}{\Sigma} b \frac{\partial A}{\partial a} \text{ or } \overset{\circ}{\Sigma} a \frac{\partial A}{\partial e} = \quad -E,$$

$$\overset{\circ}{\Sigma} c \frac{\partial A}{\partial a} \text{ or } \overset{\circ}{\Sigma} a \frac{\partial A}{\partial d} = \quad -D,$$

$$\overset{\circ}{\Sigma} d \frac{\partial A}{\partial a} \text{ or } \overset{\circ}{\Sigma} a \frac{\partial A}{\partial c} = \quad -C,$$

$$\overset{\circ}{\Sigma} e \frac{\partial A}{\partial a} \text{ or } \overset{\circ}{\Sigma} a \frac{\partial A}{\partial b} = \quad -B.$$

There are, of course, similar results in regard to the differentiation of B, C, ...; and it is readily found that the general proposition embracing them all is, '*If A_1, A_2, ... be the signed primary minors corresponding to the elements a_1, a_2, ... of a circulant of the n^{th} order, then*

$$\overset{\circ}{\Sigma} a_r \frac{\partial A_r}{\partial a_r} = (n-1) A_r,$$

$$\overset{\circ}{\Sigma} a_r \frac{\partial A_r}{\partial a_s} = \quad -A_s,$$

where s is not equal to r.'

3. Coming now to the Jacobian

$$\begin{vmatrix} \frac{\partial A_1}{\partial a_1} & \frac{\partial A_1}{\partial a_2} & \frac{\partial A_1}{\partial a_3} & \cdots & \frac{\partial A_1}{\partial a_n} \\ \frac{\partial A_2}{\partial a_1} & \frac{\partial A_2}{\partial a_2} & \frac{\partial A_2}{\partial a_3} & \cdots & \frac{\partial A_2}{\partial a_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial A_n}{\partial a_1} & \frac{\partial A_n}{\partial a_2} & \frac{\partial A_n}{\partial a_3} & \cdots & \frac{\partial A_n}{\partial a_n} \end{vmatrix},$$

and multiplying it in row-by-row fashion by

$$\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \end{vmatrix},$$

that is, by $(-1)^{\frac{1}{2}(n-1)(n-2)}.C(a_1, a_2, \ldots, a_n)$, we obtain

$$\begin{vmatrix} \overset{0}{\Sigma} a_1 \frac{\partial A_1}{\partial a_1} & \overset{0}{\Sigma} a_2 \frac{\partial A_1}{\partial a_1} & \overset{0}{\Sigma} a_3 \frac{\partial A_1}{\partial a_1} & \cdots & \overset{0}{\Sigma} a_n \frac{\partial A_1}{\partial a_1} \\ \overset{0}{\Sigma} a_1 \frac{\partial A_2}{\partial a_1} & \overset{0}{\Sigma} a_2 \frac{\partial A_2}{\partial a_1} & \overset{0}{\Sigma} a_3 \frac{\partial A_2}{\partial a_1} & \cdots & \overset{0}{\Sigma} a_n \frac{\partial A_2}{\partial a_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \overset{0}{\Sigma} a_1 \frac{\partial A_n}{\partial a_1} & \overset{0}{\Sigma} a_2 \frac{\partial A_n}{\partial a_1} & \overset{0}{\Sigma} a_3 \frac{\partial A_n}{\partial a_1} & \cdots & \overset{0}{\Sigma} a_n \frac{\partial A_n}{\partial a_1} \end{vmatrix},$$

which, in consequence of the theorem in § 2, is equal to

$$\begin{vmatrix} (n-1)A_1 & -A_n & -A_{n-1} & \cdots & -A_2 \\ (n-1)A_2 & -A_1 & -A_n & \cdots & -A_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (n-1)A_n & -A_{n-1} & -A_{n-2} & \cdots & -A_1 \end{vmatrix},$$

and therefore

$$= (-1)^{n-1}.(n-1).\begin{vmatrix} A_1 & A_n & A_{n-1} & \cdots & A_2 \\ A_2 & A_1 & A_n & \cdots & A_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_n & A_{n-1} & A_{n-2} & \cdots & A_1 \end{vmatrix},$$

$$=(-1)^{n-1}.(n-1).C(A_1, A_2, \dots, A_n),$$

$$=(-1)^{n-1}.(n-1).\{C(a_1, a_2, \dots, a_n)\}^{n-1}.$$

On dividing both sides by $(-1)^{\frac{1}{2}(n-1)(n-2)}\, C(a_1, a_2, \dots, a_n)$ we obtain the desired result: *The Jacobian of the primary minors of the circulant of the* n^{th} *order,* C_n, *is equal to*

$$(-1)^{\frac{1}{2}n(n-1)}.(n-1).\{C_n\}^{n-2}.$$

4. Since the differential-quotient of an n-line circulant with respect to any one of its elements is n times the signed primary minor corresponding to that element, it follows that the Hessian of the circulant is n^n times the Jacobian of the primary minors, and thus equals

$$(-1)^{\frac{1}{2}n(n-1)}.(n-1)\,n^n.\{C_n\}^{n-2}.$$

5. A result obtainable from the differentiations in § 2 is worth noting, although not needed in the present connection, namely, that the sum of the first differential-quotients of A, $\Sigma\dfrac{\partial A}{\partial a}$ say,

$$= \begin{vmatrix} 1 & b & c & d \\ 1 & a & b & c \\ 1 & e & a & b \\ 1 & d & e & a \end{vmatrix} + \begin{vmatrix} a & 1 & c & d \\ e & 1 & b & c \\ d & 1 & a & b \\ c & 1 & e & a \end{vmatrix} + \begin{vmatrix} a & b & 1 & d \\ e & a & 1 & c \\ d & e & 1 & b \\ c & d & 1 & a \end{vmatrix} + \begin{vmatrix} a & b & c & 1 \\ e & a & b & 1 \\ d & e & a & 1 \\ c & d & e & 1 \end{vmatrix},$$

$$= \begin{vmatrix} . & 1 & 1 & 1 & 1 \\ 1 & a & b & c & d \\ 1 & e & a & b & c \\ 1 & d & e & a & b \\ 1 & c & d & e & a \end{vmatrix},$$

and therefore *is equal to the sum of the primary minors of* A.

Capetown, S.A.,
30*th* *June*, 1906.

THE GENERAL SOLUTION OF LAPLACE'S EQUATION IN n DIMENSIONS.

By *G. N. Watson*, Scholar of Trinity College, Cambridge.

It was shown in 1902, by Dr. Whittaker,* that any solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

could be expressed in the form

$$V = \int_0^{2\pi} f(x \cos t + y \sin t + iz,\ t)\, dt;$$

and that any solution of the equation

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \frac{\partial^2 V}{\partial x_3^2} + \frac{\partial^2 V}{\partial x_4^2} = 0$$

could be expressed in the form

$$V = \int_0^{2\pi}\int_0^{2\pi} F(x_1 \cos t \cos u + x_2 \cos t \sin u + x_3 \sin t + ix_4,\ t,\ u)\, dt du;$$

where f and F are arbitrary functions.

It is natural to enquire whether the corresponding equation for n dimensions has a solution of a similar type. The proof that such a solution exists is given in this paper: it is of a similar nature to the proofs given by Dr. Whittaker for the cases given above, in which $n = 3$ and 4 respectively. Like them it depends on establishing the linear independence of a set of polynomials, but it is more difficult to prove their independence in the general case of n dimensions.

The equation, the solution of which will be considered, is

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} + \ldots + \frac{\partial^2 V}{\partial x_n^2} = 0 \qquad\qquad (1).$$

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* *Mathematische Annalen*, Vol. 57, pp. 333–355.

In the neighbourhood of any ordinary point, which we may take as origin of coordinates, V can, by Taylor's theorem, be expanded in the form

$$V = a + b_1x_1 + b_2x_2 + \ldots + b_nx_n + c_{11}x_1^2 + c_{12}x_1x_2 + \ldots + d_{111}x_1^3 + \ldots.$$

Substituting this expansion in (1) and equating to zero the coefficients of the various powers of $x_1, x_2, \ldots, x_n$, we obtain an infinite number of linear relations between the coefficients $a, b_1, b_2, \ldots, b_n, c_{11}, \ldots$.

Now consider the homogeneous polynomial of the m^{th} degree in the n variables $x_1, x_2, \ldots, x_n$.

The polynomial in which the coefficient of each term is unity is the coefficient of y^m in the expansion of

$$\frac{1}{(1-x_1y)(1-x_2y)\ldots(1-x_ny)},$$

therefore the number of terms in the homogeneous polynomial of the m^{th} degree in the n variables $x_1, x_2, \ldots, x_n$ is the coefficient of y^m in the expansion of $(1-y)^{-n}$; *i.e.* it is $\dfrac{(n+m-1)!}{m!\,(n-1)!}$.

If we apply the operator $\dfrac{\partial^2}{\partial x_1^2} + \ldots + \dfrac{\partial^2}{\partial x_n^2}$ to this polynomial, we obtain a polynomial of degree $m-2$, in which all possible terms occur; *i.e.* there are $\dfrac{(n+m-3)!}{(m-2)!\,(n-1)!}$ terms in it.

Hence if the original polynomial is a solution of (1), there are $\dfrac{(n+m-3)!}{(m-2)!\,(n-1)!}$ relations between its coefficients.

Therefore there are $\dfrac{(n+m-1)!}{m!\,(n-1)!} - \dfrac{(n+m-3)!}{(m-2)!\,(n-1)!}$, and no more, linearly independent polynomials of degree m which are solutions of (1).

Next consider the expression

$$V_m = \int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} \{x_1 \sin t_1 \sin t_2 \ldots \sin t_{n-2} + x_2 \sin t_1 \ldots \sin t_{n-3} \cos t_{n-2}$$

$$+ x_3 \sin t_1 \ldots \sin t_{n-4} \cos t_{n-3} + \ldots + x_{n-2} \sin t_1 \cos t_2 + x_{n-1} \cos t_1 + ix_n\}^m$$

$$\times \prod_{r=1}^{r=n-2} \left\{ \begin{matrix}\sin\\ \cos\end{matrix} (\beta_1 + \beta_2 + \ldots + \beta_{n-r})\, t_r \right\} dt_1\, dt_2 \ldots dt_{n-2},$$

the coefficient of x_r being $\sin t_1 \sin t_2 \ldots \sin t_{n-r-1} \cos t_{n-r}$, where $r = 2, 3, \ldots, n-1$.

For brevity we shall write this integral in the form

$$V_m = \int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi} u^m\, W\, dt_1\, dt_2 \ldots dt_{n-2},$$

W denoting $\prod\limits_{r=1}^{r=n-2} \begin{Bmatrix}\sin\\ \cos\end{Bmatrix} (\beta_1 + \beta_2 + \ldots + \beta_{n-r})\, t_r \Big\}$.

V_m is a polynomial of degree m in $x_1, x_2, \ldots, x_n$, and it is also a solution of (1): for if we apply the operator $\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ to $(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^m$, the result is

$$m(m-1)(a_1^2 + a_2^2 + \ldots + a_n^2)(a_1x_1 + a_2x_2 + \ldots + a_nx_n)^{m-2},$$

and it is easy to verify that if

$$a_1 = \sin t_1 \sin t_2 \ldots \sin t_{n-3} \sin t_{n-2},$$

$$a_2 = \sin t_1 \sin t_2 \ldots \sin t_{n-3} \cos t_{n-2},$$

$$a_3 = \sin t_1 \sin t_2 \ldots \sin t_{n-4} \cos t_{n-3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$a_{n-2} = \sin t_1 \cos t_2,$$

$$a_{n-1} = \cos t_1,$$

$$a_n = i,$$

then $a_1^2 + a_2^2 + \ldots + a_n^2 = 0$.

We proceed to show that it is possible to obtain a set of $\frac{(n+m-1)!}{m!\,(n-1)!} - \frac{(n+m-3)!}{(m-2)!\,(n-1)!}$ linearly independent polynomials from V_m by assigning suitable sets of values to $\beta_1, \beta_2, \ldots, \beta_n$ and suitably choosing sines and cosines where there is ambiguity.

Take $\beta_1, \beta_2, \ldots, \beta_{n-1}$ to be positive integers or zero.

If $\beta_1 + \beta_2 + \ldots + \beta_{n-1} > m$, $V_m = 0$.

In the course of the proof, we shall make use of the following lemma:

Lemma. A set of polynomial functions of $x_1, x_2, \ldots, x_n$, which may be termed $X_1, X_2, \ldots, X_n$, are linearly independent if each function X_r contains a term which is not present in any of the preceding $X_1, X_2, \ldots, X_{r-1}$.

Also the following integrals are required.

If p, q, r be integers,

$$\int_0^{2\pi} \sin^p t \cos^q t \begin{matrix}\sin\\ \cos\end{matrix} rt\, dt = 0, \text{ if } r > p + q \ldots\ldots (2),$$

$$\int_0^{2\pi} \sin^p t \cos^q t \sin (p+q)\, t\, dt = 0, \text{ if } p \text{ be even} \ldots\ldots (3),$$

$$\neq 0, \text{ if } p \text{ be odd} \ldots\ldots (4),$$

$$\int_0^{2\pi} \sin^p t \cos^q t \cos (p+q)\, t\, dt = 0, \text{ if } p \text{ be odd} \ldots\ldots (5),$$

$$\neq 0, \text{ if } p \text{ be even} \ldots\ldots (6).$$

Now consider the general term in V_m. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive integers, or zero, satisfying the relation

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = m,$$

the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ in V_m is a numerical multiple of

$$\int_0^{2\pi}\int_0^{2\pi} \ldots \int_0^{2\pi} (\sin^{\alpha_1+\alpha_2+\ldots+\alpha_{n-2}} t_1 \cos^{\alpha_{n-1}} t_1)(\sin^{\alpha_1+\alpha_2+\ldots+\alpha_{n-3}} t_2 \cos^{\alpha_{n-2}} t_2) \ldots$$

$$\ldots (\sin^{\alpha_1} t_{n-2} \cos^{\alpha_2} t_{n-2}) \times \begin{matrix}\sin\\ \cos\end{matrix} (\beta_1+\beta_2+\ldots+\beta_{n-1}) t_1 \begin{matrix}\sin\\ \cos\end{matrix} (\beta_1+\beta_2+\ldots+\beta_{n-2}) t_2 \ldots$$

$$\times \begin{matrix}\sin\\ \cos\end{matrix} (\beta_1 + \beta_2)\, t_{n-2}\, dt_1\, dt_2 \ldots dt_{n-2};$$

therefore the term $x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$ (where $\beta_1 + \beta_2 + \ldots + \beta_n = m$) is present in V_m provided that in the expression

$$\begin{matrix}\sin\\ \cos\end{matrix} (\beta_1 + \beta_2 + \ldots + \beta_{n-r})\, t_r$$

we choose sine or cosine according as $\beta_1 + \beta_2 + \ldots + \beta_{n-r-1}$ is odd or even, for $r = 0, 1, 2, \ldots, n-2$.

This follows at once from (4) and (6) and determines the ambiguities of sine and cosine in W.

But the term $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ is not present in V_m if any one of the following inequalities hold:

$$\beta_1 + \beta_2 > \alpha_1 + \alpha_2, \quad \beta_1 + \beta_2 + \beta_3 > \alpha_1 + \alpha_2 + \alpha_3, \quad \ldots,$$

$$\beta_1 + \beta_2 + \ldots + \beta_{n-1} > \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}.$$

This follows from (2).

Again, so long as $\beta_1 = 0$, there is no term in V_m which contains an odd power of x_1. For in order that $x_2^{\beta_2} \ldots x_n^{\beta_n}$ may be present in V_m, of the two $\begin{matrix}\sin\\ \cos\end{matrix} (\beta_1 + \beta_2)\, t_{n-2}$, we must choose $\cos (\beta_1 + \beta_2)\, t_{n-2}$, and then by (5), if α_1 be odd, the coefficient of $x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ vanishes.

Similarly by (3), so long as $\beta_1 = 1$, there is no term in V_m which contains an even power of x_1.

Consider now in order the following scheme of sets of values for

β_1,	β_2,	β_3, ...,	β_{n-3},	β_{n-2},	β_{n-1},	β_n,
0	0	0	0	0	0	m
0	0	0	0	0	1	$m-1$
0	0	0	0	0	2	$m-2$
			....	....	....	....
			0	0	m	0
			0	1	0	$m-1$
			0	1	1	$m-2$
			....	....	....	....
			0	1	$m-1$	0
			0	2	0	$m-2$
			0	2	1	$m-3$
			....	....	....	....
			0	2	$m-2$	0
			....	....	....	....
			....	....	....	....
			0	$m-1$	0	1
			0	$m-1$	1	0
			0	m	0	0
			1	0	0	$m-1$
			1	0	1	$m-2$
			....	....	....	....
			....	....	....	....
			1	$m-1$	0	0
			2	0	0	$m-2$
			....	....	....	....
			....	....	....	....
0	0	0	m	0	0	0
....	....	....	....	....	....	....
....	....	....	....	....	....	....
0	m	0	0	0	0	0
═══	═══	═══	═══	═══	═══	═══
1	0	0	0	0	0	$m-1$
....	....	....	....	....	....	....
1	$m-1$	0	0	0	0	0

The portion of the scheme below the double lines $=$ is formed similarly to the portion above except that $\beta_1 = 1$ instead of 0 throughout the lower portion.

If now we compare any set of values for $\beta_1, \beta_2, \ldots, \beta_n$ (down to the double line $=$) with a previous set, which may be termed $\gamma_1, \gamma_2, \ldots, \gamma_n$, one of the relations

$$\beta_1+\beta_2>\gamma_1+\gamma_2,\ \beta_1+\beta_2+\beta_3>\gamma_1+\gamma_2+\gamma_3,\ \ldots,$$
$$\beta_1+\beta_2+\ldots+\beta_{n-1}>\gamma_1+\gamma_2+\ldots+\gamma_{n-1}$$

will hold; for if β_r be the first β which is not equal to γ_r, then $\beta_r>\gamma_r$.

Hence for any one set of the β's there is a term $x_1^{\gamma_1}x_2^{\gamma_2}\ldots x_n^{\gamma_n}$ in the corresponding V_m, which is not present in the V_m corresponding to any subsequent sets of β's down to the double line.

The same property holds for the scheme of sets of β's below the double line, in all of which $\beta_1=1$.

Again, no term which occurs in the V_m corresponding to a set of β's above the double line can occur in any V_m corresponding to a set of β's below the double line; for it has been proved that, when $\beta_1=0$, V_m contains no terms involving odd powers of x_1, and that, when $\beta_1=1$, V_m contains no term involving even powers of x_1. Therefore the V_m which corresponds to each set of β's in the complete scheme contains a term which is not present in the V_m corresponding to any subsequent set of β's.

Therefore, if we re-arrange the scheme of sets of β's in the reverse order, the V_m which corresponds to each set of β's contains a term which is not present in the V_m corresponding to any preceding set of β's in the re-arranged scheme.

Therefore, by the lemma, all the V_m's corresponding to sets of β's in the scheme are linearly independent.

The number of sets of β's in the scheme is the number of positive integral solutions (including zero) of the equation

$$\beta_2+\beta_3+\ldots+\beta_n=m$$

plus the number of positive integral solutions (including zero) of the equation

$$\beta_2+\beta_3+\ldots+\beta_n=m-1.$$

Therefore the number of linearly independent V_m's obtained

$$=\frac{(n+m-2)!}{m!\,(n-2)!}+\frac{(n+m-3)!}{(m-1)!\,(n-2)!},$$

and it may be easily verified that this number

$$=\frac{(n+m-1)!}{m!\,(n-1)!}-\frac{(n+m-3)!}{(m-2)!\,(n-1)!}.$$

Hence, by choosing different forms for W,

$$\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} u^m W dt_1 dt_2 \ldots dt_{n-2}$$

can be made to represent a complete set of linearly independent polynomials of degree m which are solutions of (1).

Therefore every polynomial of degree m which satisfies (1) is of the form

$$\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} u^m \Sigma A_{\beta_1,\beta_2,\ldots,\beta_{n-1}}$$

$$\times \prod_{r=1}^{r=n-2} {}^{\sin}_{\cos}(\beta_1+\beta_2+\ldots+\beta_{n-r})\, t_r \,.\, dt_1 dt_2 \ldots dt_{n-2},$$

the summation extending over all values of $\beta_1, \beta_2, \ldots, \beta_{n-1}$, and the A's being arbitrary constants.

Now every expression of this type is comprised in the expression

$$V_m' = \int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} u^m f_m(t_1, t_2, \ldots, t_{n-2})\, dt_1 dt_2 \ldots dt_{n-2},$$

where f_m is an arbitrary function, finite for all values of $t_1, t_2, \ldots, t_{n-2}$ in the open interval 0 to 2π.

And if f_m be an arbitrary function subject to this restriction, V_m' is always a solution of (1).

Therefore if the origin be not a branch point nor a pole of V, the general solution of (1) throughout a certain region including the origin is of the form

$$V' = \sum_{m=0}^{\infty} a_m \int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} u^m f_m(t_1, t_2, \ldots, t_{n-2})\, dt_1 dt_2 \ldots dt_{n-2},$$

the a's being arbitrary constants.

$$= \int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} f(u, t_1, t_2, \ldots, t_{n-2})\, dt_1 dt_2 \ldots dt_{n-2},$$

where $f(u, t_1, t_2, \ldots, t_{n-2})$ is an arbitrary function subject to the restriction that for all values of $t_1, t_2, \ldots, t_{n-2}$ in the open

interval 0 to 2π, f is finite throughout the region and can be expanded in the form $\sum_{m=0}^{\infty} a_m u^m$, the a's being functions of $t_1, t_2, \ldots, t_{n-2}$.

And if f be subject to this restriction, V' is always a solution of (1).

Next suppose that the origin may be a branch point or a pole of V.

Then V can be represented throughout its region of existence by a set of expansions in integral power series about a set of points which may be called $b_1, b_2, \ldots, b_n$; $c_1, c_2, \ldots, c_n$;...: these points are not branch points nor poles of V.

Let the values of u at these points be $u_b, u_c, \ldots$.

Then, since the equation (1) can be written in the forms

$$\frac{\partial^2 V}{\partial (x_1 - b_1)^2} + \frac{\partial^2 V}{\partial (x_2 - b_2)^2} + \ldots \frac{\partial^2 V}{\partial (x_n - b_n)^2} = 0,$$

$$\frac{\partial^2 V}{\partial (x_1 - c_1)^2} + \frac{\partial^2 V}{\partial (x_2 - c_2)^2} + \ldots \frac{\partial^2 V}{\partial (x_n - c_n)^2} = 0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

it follows that the general solution of (1) can be completely represented by a set of expressions of the form

$$\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} f_b(u - u_b, t_1, t_2, \ldots, t_{n-2})\, dt_1 dt_2 \ldots dt_{n-2},$$

$$\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} f_c(u - u_c, t_1, t_2, \ldots, t_{n-2})\, dt_1 dt_2 \ldots dt_{n-2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where $f_b, f_c, \ldots$ are arbitrary functions subject to the same restriction as f.

Since the quantities $u_b, u_c, \ldots$ do not contain $x_1, x_2, \ldots, x_n$ but only $t_1, t_2, \ldots, t_{n-2}$ and certain constants $b_1, b_2, \ldots, b_n$, $c_1, c_2, \ldots, c_n, \ldots$, this set of expressions may be represented by the expression

$$\int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} F(u, t_1, t_2, \ldots, t_{n-2})\, dt_1 dt_2 \ldots dt_{n-2},$$

where F is an arbitrary function which is capable of being differentiated twice with respect to u; *i.e.* every solution of (1)

is of the form

$$V=\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}F(u,\,t_1,\,t_2,\,\dots,\,t_{n-2})\,dt_1dt_2\dots dt_{n-2}.$$

And this expression is always a solution of (1), if F can be differentiated twice with respect to u.

That is to say, the general solution of

$$\frac{\partial^2 V}{\partial x_1^{\,2}}+\frac{\partial^2 V}{\partial x_2^{\,2}}+\dots+\frac{\partial^2 V}{\partial x_n^{\,2}}=0$$

is

$$V=\int_0^{2\pi}\int_0^{2\pi}\dots\int_0^{2\pi}F\{x_1\sin t_1\sin t_2\dots\sin t_{n-2}$$

$$+\sum_{r=2}^{r=n-2}(x_r\sin t_1\sin t_2\dots\sin t_{n-r-1}\cos t_{n-r})+x_{n-1}\cos t_1+ix_n,$$

$$t_1,\,t_2,\,\dots,\,t_{n-2}\}\,dt_1dt_2\dots dt_{n-2},$$

where F is an arbitrary function which is capable of being differentiated twice with respect to $x_1,\,x_2,\,\dots,\,x_n$.

DEDUCTION OF FORMULÆ IN SPHERICAL TRIGONOMETRY FROM THOSE OF PLANE TRIGONOMETRY.

By *H. G. Dawson, M.A.*

Let the sides AC, AB of a spherical triangle meet again at A'; the sides BC, BA at B'; and CA, CB at C'; then, if from C we project the spherical figure on the tangent plane at C, the great circle $A'BAB'$ projects into a circle and the great circles $B'CBC'$, $A'CAC'$ project into straight lines, and we obtain a plane figure consisting of a circle X and two chords.

Now let the sides and angles of the spherical triangle be a, b, c, A, B, C, and let the projections of A, B, A', B' be

$\alpha, \beta, \alpha', \beta'$. We have

$$C\beta = 2k\tan\tfrac{1}{2}a, \quad C\alpha = 2k\tan\tfrac{1}{2}b,$$

$$C\beta' = 2k\cot\tfrac{1}{2}a, \quad C\alpha' = 2k\cot\tfrac{1}{2}b.$$

$2k$ being the radius of the sphere.

Hence

$$\alpha\beta^2 = 4k^2\left(\tan^2\tfrac{1}{2}a + \tan^2\tfrac{1}{2}b - 2\tan\tfrac{1}{2}a\tan\tfrac{1}{2}b \cdot \frac{\cos c - \cos a\cos b}{\sin a\sin b}\right)$$

$$= \frac{\sin^2\frac{1}{2}c}{\cos^2\frac{1}{2}a\cos^2\frac{1}{2}b},$$

or $$\alpha\beta = 2k\sin\tfrac{1}{2}c/\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b,$$

similarly $$\alpha'\beta = 2k\cos\tfrac{1}{2}c/\cos\tfrac{1}{2}a\sin\tfrac{1}{2}b,$$

$$\alpha\beta' = 2k\cos\tfrac{1}{2}c/\sin\tfrac{1}{2}a\cos\tfrac{1}{2}b,$$

and $$\alpha'\beta' = 2k\sin\tfrac{1}{2}c/\sin\tfrac{1}{2}a\sin\tfrac{1}{2}b,$$

the latter three equations being easily deduced from the first by changing a, b to their supplements.

Now let the tangent at α to the circle X cut the tangent at β in the point P, and the tangent at β' in the point S, and let the tangent at α' cut the tangents at β, β' in the points Q, R respectively, then

$$P\alpha C = A, \quad P\beta C = B, \quad C\beta Q = \pi - B,$$

$$C\alpha' Q = A, \quad C\alpha' R = \pi - A, \quad C\beta' R = \pi - B,$$

$$C\beta' S = B, \quad C\alpha S = \pi - A,$$

and if we denote the angles $C\alpha\beta$, $C\beta\alpha$, $\alpha\beta'\beta$, $\beta'\beta\alpha'$ by θ, ϕ, ψ, χ, then we easily see that

$$2\psi = A + B + C - \pi = E,$$

therefore $$\psi = \tfrac{1}{2}E \text{ or } S - \tfrac{1}{2}\pi;$$

$$\theta = A - \psi,$$

therefore $$\theta = \tfrac{1}{2}\pi - (S - A) \text{ or } A - \tfrac{1}{2}E;$$

$$\phi = B - \psi,$$

therefore $$\phi = \tfrac{1}{2}\pi - (S - B) \text{ or } B - \tfrac{1}{2}E;$$

$$\chi = C - \psi;$$

therefore $$\chi = \tfrac{1}{2}\pi - (S - C) \text{ or } C - \tfrac{1}{2}E.$$

We can now directly deduce most of the formulæ of the spherical triangle directly from the ordinary properties of the triangles in this projected figure.

Napier's analogies.

Apply to the triangle $C\alpha\beta$ the formula

$$\tan\tfrac{1}{2}(A - B) = \frac{a-b}{a+b}\cot\tfrac{1}{2}C,$$

therefore $$\tan\tfrac{1}{2}(\theta - \phi) = \frac{C\beta - C\alpha}{C\beta + C\alpha}\cot\tfrac{1}{2}C,$$

or $$\tan\tfrac{1}{2}(A - B) = \frac{\tan\tfrac{1}{2}a - \tan\tfrac{1}{2}b}{\tan\tfrac{1}{2}a + \tan\tfrac{1}{2}b}\cot\tfrac{1}{2}C.$$

Hence $$\tan\tfrac{1}{2}(A - B) = \frac{\sin\tfrac{1}{2}(a-b)}{\sin\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C.$$

Turn to the triangle $BC\alpha'$ and again apply the same formula, and we have

$$\tan\tfrac{1}{2}(\chi - \psi) = \frac{C\alpha' - C\beta}{C\alpha' + C\beta}\cot\tfrac{1}{2}(\pi - C) = \frac{\cot\tfrac{1}{2}b - \tan\tfrac{1}{2}a}{\cot\tfrac{1}{2}b + \tan\tfrac{1}{2}a}\tan\tfrac{1}{2}C,$$

or $$\cot\tfrac{1}{2}(A + B) = \frac{\cos\tfrac{1}{2}(a+b)}{\cos\tfrac{1}{2}(a-b)}\tan\tfrac{1}{2}C.$$

Hence $$\tan\tfrac{1}{2}(A - B) = \frac{\cos\tfrac{1}{2}(a-b)}{\cos\tfrac{1}{2}(a+b)}\cot\tfrac{1}{2}C,$$

and the remaining pair of formulæ might be obtained from the triangle $\alpha C\beta$, $\alpha' C\beta'$.

Gauss's analogies.

These are obtained in a similar fashion, we apply the pair of plane formulæ

$$\cos\tfrac{1}{2}(A-B)=\frac{a+b}{c}\sin\tfrac{1}{2}C,$$

$$\sin\tfrac{1}{2}(A-B)=\frac{a-b}{c}\cos\tfrac{1}{2}C$$

to each of the triangles $\alpha C\beta$, &c.

Thus, applying the first of these formulæ to the triangle $\alpha C\beta$,

$$\cos\tfrac{1}{2}(\theta-\phi)=\frac{C\beta+C\alpha}{\alpha\beta}\sin\tfrac{1}{2}C,$$

therefore $$\cos\tfrac{1}{2}(A-B)=\frac{\tan\frac{1}{2}a+\tan\frac{1}{2}b}{\sin\frac{1}{2}c\sec\frac{1}{2}a\sec\frac{1}{2}b}\sin\tfrac{1}{2}C,$$

or $$\cos\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\sin\tfrac{1}{2}C.$$

Similarly the other formula gives

$$\sin\tfrac{1}{2}(\theta-\phi)=\frac{C\beta-C\alpha}{\alpha\beta}\sin\tfrac{1}{2}C,$$

or

$$\sin\tfrac{1}{2}(A-B)=\frac{\tan\frac{1}{2}a-\tan\frac{1}{2}b}{\sin\frac{1}{2}c\sec\frac{1}{2}a\sec\frac{1}{2}b}\sin\tfrac{1}{2}C=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\sin\tfrac{1}{2}C.$$

The other Gaussian analogies can be obtained in the same manner from the triangles $\alpha' C\beta$, &c.

If the diameter of the circle X be D, then, since

$$\alpha\alpha'=D\sin(\phi+\chi)=D\sin A,$$

and $$\beta\beta'=D\sin(\phi+\psi)=D\sin B,$$

therefore

$$\sin A:\sin B::\alpha\alpha':\beta\beta'::\tan\tfrac{1}{2}b+\cot\tfrac{1}{2}b:\tan\tfrac{1}{2}a+\cot\tfrac{1}{2}a,$$

or $$\sin A:\sin B::\sin a:\sin b,$$

and $$D=4k\operatorname{cosec}a\operatorname{cosec}B.$$

To prove $\cot b \sin a = \cos a \cos C + \cot B \sin C$.

Let O be the centre of the circle X, and OM, ON the perpendiculars on $\alpha\alpha'$, $\beta\beta'$.

We have $MC = \frac{1}{2}(2k \tan \frac{1}{2}b - 2k \cot \frac{1}{2}b) = -2k \cot b$,

$$NC = -2k \cot a.$$

Now project the sides of the triangle OCN on CM, therefore

$$CM = CN \cos C + ON \sin C,$$

therefore

$$-2k \cot b = -2k \cot a \cos C + N\beta' \cot \beta\alpha\beta' \sin C,$$

or

$$\cot b = \cot a \cos C + \cot B \sin C \operatorname{cosec} a,$$

since $\beta\alpha\beta' = \pi - B$ and $N\beta' = 2k \operatorname{cosec} a$.

The sides of the plane triangle $\alpha C\beta$ being $2k \tan \frac{1}{2}b$, $2k \tan \frac{1}{2}a$, $2k \sin \frac{1}{2}c / \cos \frac{1}{2}a \cos \frac{1}{2}b$, and the opposite angles θ, ϕ, C, or $\frac{1}{2}\pi - (S-A)$, $\frac{1}{2}\pi - (S-B)$, C.

The formulæ for $\cos \frac{1}{2}C$, $\sin \frac{1}{2}C$, $\tan \frac{1}{2}C$ are deduced by applying to this triangle the *corresponding* formulæ in plane trigonometry, thus

$$\cos^2 \tfrac{1}{2}C = \frac{\left\{\begin{array}{l}(\tan \frac{1}{2}a + \tan \frac{1}{2}b + \sin \frac{1}{2}c \sec \frac{1}{2}a \sec \frac{1}{2}b) \\ \qquad (\tan \frac{1}{2}a + \tan \frac{1}{2}b - \sin \frac{1}{2}c \sec \frac{1}{2}a \sec \frac{1}{2}b)\end{array}\right\}}{4 \tan \frac{1}{2}a \tan \frac{1}{2}b}$$

$$= \frac{\{\sin \frac{1}{2}(a+b) + \sin \frac{1}{2}c\}\{\sin \frac{1}{2}(a+b) - \sin \frac{1}{2}c\}}{\sin a \sin b}.$$

$$= \frac{\sin^2 \frac{1}{2}(a+b) - \sin^2 \frac{1}{2}c}{\sin a \sin b} = \frac{\sin s . \sin (s-c)}{\sin a \sin b},$$

and so on.

We have also

$$\tan \tfrac{1}{2}a / \cot \tfrac{1}{2}a = \frac{C\beta}{C\beta'} = \frac{C\beta}{C\alpha} \cdot \frac{C\alpha}{C\beta'} = \frac{\sin \theta}{\sin \phi} \cdot \frac{\sin \psi}{\sin \chi}$$

$$= -\frac{\cos (S-A) \cos S}{\cos (S-B) \cos (S-C)},$$

or

$$\tan \tfrac{1}{2}a = \sqrt{\left\{\frac{-\cos S . \cos (S-A)}{\cos (S-B) \cos (S-C)}\right\}},$$

and so on.

Corresponding to the formulæ $a = b\cos C + c\cos B$, &c., we have

$$\sin\tfrac{1}{2}c/\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b = \tan\tfrac{1}{2}b\sin(S-A) + \tan\tfrac{1}{2}a\sin(S-B);$$

that is,

$$\sin\tfrac{1}{2}c = \sin\tfrac{1}{2}a\cos\tfrac{1}{2}b\sin(S-B) + \sin\tfrac{1}{2}b\cos\tfrac{1}{2}a\sin(S-A), \text{ \&c.}$$

Cagnoli's theorem.

We have $\alpha\beta = D\sin\psi = D\sin\tfrac{1}{2}E,$

therefore $2k\sin\tfrac{1}{2}c/\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b = \dfrac{4k}{\sin a\sin B}\sin\tfrac{1}{2}E.$

Therefore $\sin\tfrac{1}{2}E = \dfrac{\sin a\sin\tfrac{1}{2}c\sin B}{2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b} = \dfrac{n}{2\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c},$

where $n^2 = \sin s.\sin(s-a)\sin(s-b)\sin(s-c).$

Lhuilier's theorem.

The functions of $\tfrac{1}{4}E$ are deduced from the triangle $\alpha\alpha'\beta$, its sides are proportional to $\cos\tfrac{1}{2}a$, $\sin\tfrac{1}{2}b\sin\tfrac{1}{2}c$, $\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c$, and the angle opposite them are $\pi - A$, $\tfrac{1}{2}E$, $A - \tfrac{1}{2}E$.

Therefore

$$\cos^2\tfrac{1}{4}E = \frac{\{\cos\tfrac{1}{2}(b+c) + \cos\tfrac{1}{2}a\}\{\cos\tfrac{1}{2}(b-c) + \cos\tfrac{1}{2}a\}}{4\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c}$$

$$= \frac{\cos\tfrac{1}{2}s\cos\tfrac{1}{2}(s-a)\cos\tfrac{1}{2}(s-b)\cos\tfrac{1}{2}(s-c)}{4\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c},$$

whilst

$$\sin^2\tfrac{1}{4}E = \frac{\{\cos\tfrac{1}{2}(b-c) - \cos\tfrac{1}{2}a\}\{\cos\tfrac{1}{2}a - \cos\tfrac{1}{2}(b+c)\}}{4\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c}$$

$$= \frac{\sin\tfrac{1}{2}s\sin\tfrac{1}{2}(s-a)\sin\tfrac{1}{2}(s-b)\sin\tfrac{1}{2}(s-c)}{4\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b\cos\tfrac{1}{2}c},$$

the quotient giving Lhuilier's theorem.

To prove $\cot\tfrac{1}{2}E = \dfrac{\cot\tfrac{1}{2}a\cot\tfrac{1}{2}b + \cos C}{\sin C}.$

In the triangle $\beta C\alpha'$, the angles opposite to $C\beta$, $C\alpha'$ are $C - \tfrac{1}{2}E$, $\tfrac{1}{2}E$. Therefore

$$\sin(C-\tfrac{1}{2}E)/\sin\tfrac{1}{2}E = \sin\chi/\sin\psi = C\alpha'/C\beta = \cot\tfrac{1}{2}b/\tan\tfrac{1}{2}a = \cot\tfrac{1}{2}a\cot\tfrac{1}{2}b,$$

therefore $\cot\frac{1}{2}E.\sin C - \cos C = \cot\frac{1}{2}a\cot\frac{1}{2}b$,

therefore, &c.

Again the sides of the triangle $\beta C\alpha'$ being proportional to $\sin\frac{1}{2}a\sin\frac{1}{2}b$, $\cos\frac{1}{2}a\cos\frac{1}{2}b$, $\cos\frac{1}{2}c$, we have, applying the plane formula for the cosine of an angle of a triangle,

$$\cos(C-\tfrac{1}{2}E) = \frac{\sin^2\frac{1}{2}a\sin^2\frac{1}{2}b + \cos^2\frac{1}{2}c - \cos^2\frac{1}{2}a\cos^2\frac{1}{2}b}{2\sin\frac{1}{2}a\sin\frac{1}{2}b\cos\frac{1}{2}c}$$

$$= \frac{1-\cos^2\frac{1}{2}a - \cos^2\frac{1}{2}b + \cos^2\frac{1}{2}c}{2\sin\frac{1}{2}a\sin\frac{1}{2}b\cos\frac{1}{2}c},$$

$$\cos\tfrac{1}{2}E = \frac{\cos^2\frac{1}{2}c + \cos^2\frac{1}{2}a\cos^2\frac{1}{2}b - \sin^2\frac{1}{2}a\sin^2\frac{1}{2}b}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c}$$

$$= \frac{\cos^2\frac{1}{2}a + \cos^2\frac{1}{2}b + \cos^2\frac{1}{2}c - 1}{2\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c},$$

whilst the formula for the cosine of the half angle gives

$$\cos^2\tfrac{1}{2}(c-\tfrac{1}{2}E) = \frac{\{\cos\frac{1}{2}c + \cos\frac{1}{2}(a-b)\}\{\cos\frac{1}{2}c - \cos\frac{1}{2}(a+b)\}}{4\sin\frac{1}{2}a\sin\frac{1}{2}b\cos\frac{1}{2}c}$$

$$= \frac{\sin\frac{1}{2}s\sin\frac{1}{2}(s-c)\cos\frac{1}{2}(s-b)\cos\frac{1}{2}(s-c)}{\sin\frac{1}{2}a\sin\frac{1}{2}b\cos\frac{1}{2}c},$$

Other formulæ are equally easy to obtain from the plane figure, for instance, applying the plane formula $a = b\cos C + c\cos B$ to the triangle $C\alpha'\beta$, we get

$$\cos\tfrac{1}{2}a\cos\tfrac{1}{2}b = \cos\tfrac{1}{2}c\cos\tfrac{1}{2}E + \sin\tfrac{1}{2}a\sin\tfrac{1}{2}b\cos(\pi - C),$$

hence $\cos\frac{1}{2}E = (\cos\frac{1}{2}a\cos\frac{1}{2}b + \sin\frac{1}{2}a\sin\frac{1}{2}b\cos C)\sec\frac{1}{2}c$,

whilst from the same triangle

$$\sin\tfrac{1}{2}E : \sin C :: \tan\tfrac{1}{2}a : \cos\tfrac{1}{2}c/\sin\tfrac{1}{2}b\cos\tfrac{1}{2}a,$$

therefore $\sin\frac{1}{2}E = \sin C.\sin\frac{1}{2}a\sin\frac{1}{2}b\sec\frac{1}{2}c.$

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PROOF OF A PROPERTY OF CONICS TOUCHING GIVEN STRAIGHT LINES.

By W. W. Taylor.

PART I.

IN January, 1906, my brother, Mr. H. M. Taylor, suggested to me the probable truth of the properties:—

The centres of the six conics each of which touches 5 of six given straight lines lie on a conic: and, further, if seven straight lines be taken, the seven such conics pass through a point.

The following is a proof of the first of these propositions, an extension of the second, and a further generalization.

§ 1. Using areal coordinates the equations of four of the lines can be taken in the form

$$l\alpha \pm m\beta \pm n\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation of any conic touching these is of the form

$$u\alpha^2 + v\beta^2 + w\gamma^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

subject to the condition

$$l^2/u + m^2/v + n^2/w = 0.$$

The condition that this conic should also touch the straight line

$$l_1\alpha + m_1\beta + n_1\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is
$$l_1^2/u + m_1^2/v + n_1^2/w = 0.$$

The coordinates of the centre of (2) are given by the equations $u\alpha = v\beta = w\gamma$; therefore the centre is the intersection of the lines

$$\left.\begin{array}{l} l^2\alpha + m^2\beta + n^2\gamma\ (= x \text{ say}) = 0 \\ l_1^2\alpha + m_1^2\beta + n_1^2\gamma\ (= y \text{ say}) = 0 \end{array}\right\} \quad\ldots\ldots\ldots\ldots(4),$$

Similarly the centre of the conic touching (1) and

$$l_2\alpha + m_2\beta + n_2\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

is the intersection of the lines

$$\left.\begin{array}{r} x = l^2\alpha + m^2\beta + n^2\gamma = 0 \\ l_2^2\alpha + m_2^2\beta + n_2^2\gamma\ (= z \text{ say}) = 0 \end{array}\right\} \ldots\ldots\ldots\ldots(6).$$

Again, let us consider the conic touching the 5 lines

$$\begin{aligned} -l\alpha + m\beta + n\gamma &= 0, \\ l\alpha - m\beta + n\gamma &= 0, \\ l\alpha + m\beta - n\gamma &= 0, \\ l_1\alpha + m_1\beta + n_1\gamma &= 0, \\ l_2\alpha + m_2\beta + n_2\gamma &= 0. \end{aligned}$$

Let its equation be

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0 \ldots\ldots(7).$$

Then if capital letters are used for the minors of the discriminant, the conditions of contact are

$$Ul^2 + Vm^2 + Wn^2 + 2U'mn - 2V'nl - 2W'lm = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

whence

$$U'/l = V'/m = W'/n = (Ul^2 + Vm^2 + Wn^2)/2lmn = K \text{ say} \ldots(8),$$

$$Ul_1^2 + Vm_1^2 + Wn_1^2 + 2K(lm_1n_1 + mn_1l_1 + nl_1m_1) = 0 \ldots(9),$$

and

$$Ul_2^2 + Vm_2^2 + Wn_2^2 + 2K(lm_2n_2 + mn_2l_2 + nl_2m_2) = 0 \ldots(10).$$

Now the centre of the conic (7) is given by the equations

$$\alpha/(U + V' + W') = \beta/(V + W' + U') = \gamma/(W + U' + V') = L \text{ say},$$

therefore

$$\begin{aligned} x &= l^2\alpha + m^2\beta + n^2\gamma \\ &= \{Ul^2 + Vm^2 + Wn^2 + (V' + W')l^2 + (W' + U')m^2 + (U' + V')n^2\}L \\ &= (m + n)(n + l)(l + m)KL \text{ by (8).} \end{aligned}$$

Similarly

$$y = \{l(m_1 - n_1)^2 + m(n_1 - l_1)^2 + n(l_1 - m_1)^2\}KL,$$
$$z = \{l(m_2 - n_2)^2 + m(n_2 - l_2)^2 + n(l_2 - m_2)^2\}KL.$$

So the centre of the conic (7) is given by the equations

$$x = y/a = z/b \text{.....................} (11),$$

here

$$= \{l(m_1 - n_1)^2 + m(n_1 - l_1)^2 + n(l_1 - m_1)^2\}/\{(m+n)(n+l)(l+m)\}$$

nd

$$= \{l(m_2 - n_2)^2 + m(n_2 - l_2)^2 + n(l_2 - m_2)^2\}/\{(m+n)(n+l)(l+m)\}.$$

Let a_1, b_1; a_2, b_2; a_3, b_3 be the values assumed by a, b when he signs of l, m, n respectively are changed.

Then $$x = y/a_1 = z/b_1, \text{ \&c.} \text{.....................} (12)$$

re the equations of the centres of the conics which touch 5 of he original six straight lines, omitting $-l\alpha + m\beta + n\gamma = 0$, &c. espectively.

The equations of a conic through the two points (4), (6) an be written

$$x(Ax + By + Cz) + yz = 0 \text{.............} (13).$$

If the points (12) are on this conic

$$A + Ba_1 + Cb_1 + a_1 b_1 = 0,$$

$$A + Ba_2 + Cb_2 + a_2 b_2 = 0,$$

$$A + Ba_3 + Cb_3 + a_3 b_3 = 0,$$

and the equation of the conic can be written

$$\begin{vmatrix} x^2, & xy, & xz, & yz \\ 1, & a_1, & b_1, & a_1 b_1 \\ 1, & a_2, & b_2, & a_2 b_2 \\ 1, & a_3, & b_3, & a_3 b_3 \end{vmatrix} = 0 \text{..............} (14).$$

The condition for this to pass also through the point (11) is

$$\begin{vmatrix} 1, & a, & b, & ab \\ 1, & a_1, & b_1, & a_1 b_1 \\ 1, & a_2, & b_2, & a_2 b_2 \\ 1, & a_3, & b_3, & a_3 b_3 \end{vmatrix} = 0.$$

Now a, a_1, a_2, a_3 only involve differences of l_1, m_1, n_1; therefore the condition is satisfied always, if it is satisfied for any straight line parallel to $l_1\alpha + m_1\beta + n_1\gamma = 0$, but a straight line can always be drawn in a given direction to touch a conic, and so the condition is always satisfied.

§ 2. The equation (14) can (after freeing it of irrelevant factors) be reduced to the form

$$x^2\Sigma(m_1n_2-m_2n_1)\,\Sigma[l^2(m_1-n_1)(m_2-n_2)\{(n_1-l_1)(n_2-l_2)-(l_1-m_1)(l_2-m_2)\}]$$
$$-\,xy\Sigma\,[l^4\,(m_1-n_1)\,(m_2-n_2)^3$$
$$+m^2n^2\{(n_1-l_1)(l_2-m_2)^3+(l_1-m_1)(n_2-l_2)^3-(m_1-n_1)(m_2-n_2)(n_2-l_2)(l_2-m_2)\}]$$
$$+\,xz\Sigma\,[l^4\,(m_2-n_2)\,(m_1-n_1)^3$$
$$+m^2n^2\{(n_2-l_2)(l_1-m_1)^3+(l_2-m_2)(n_1-l_1)^3-(m_2-n_2)(m_1-n_1)(n_1-l_1)(l_1-m_1)\}]$$
$$-\,yz\Sigma(m_1n_2-m_2n_1)\,(m^2-n^2)\,(n^2-l^2)\,(l^2-m^2)=0.$$

§ 3. By projection we can see that "the poles of any seventh straight line with regard to the six conics, each of which touches five of six given straight lines, lie on a conic," and by reciprocation "the polars of any seventh point with regard to the six conics, each of which passes through 5 of six given points touch a conic."

§ 14. The condition that the conic (14) should break up into two straight lines is

$$(a_2-a_3)\,(a_3-a_1)\,(a_1-a_2)\,(b_2-b_3)\,(b_3-b_1)\,(b_1-b_2)=0.$$

For its equation will resolve into factors if

$$\begin{vmatrix} a_1, & b_1, & a_1b_1 \\ a_2, & b_2, & a_2b_2 \\ a_3, & b_3, & a_3b_3 \end{vmatrix}\begin{vmatrix} 1, & a_1, & b_1 \\ 1, & a_2, & b_2 \\ 1, & a_3, & b_3 \end{vmatrix} = \begin{vmatrix} b_1, & a_1b_1, & 1 \\ b_2, & a_2b_2, & 1 \\ b_3, & a_3b_3, & 1 \end{vmatrix}\begin{vmatrix} a_1b_1, & 1, & a_1 \\ a_2b_2, & 1, & a_2 \\ a_3b_2, & 1, & a_3 \end{vmatrix},$$

That is, if

$$\begin{vmatrix} 1/b_1, & 1/a_1, & 1 \\ 1/b_2, & 1/a_2, & 1 \\ 1/b_3, & 1/a_3, & 1 \end{vmatrix}\begin{vmatrix} 1, & a_1, & b_1 \\ 1, & a_2, & b_2 \\ 1, & a_3, & b_3 \end{vmatrix} + \begin{vmatrix} 1, & a_1, & 1/b_1 \\ 1, & a_2, & 1/b_2 \\ 1, & a_3, & 1/b_3 \end{vmatrix}\begin{vmatrix} 1/a_1, & b_1, & 1 \\ 1/a_2, & b_2, & 1 \\ 1/a_3, & b_3, & 1 \end{vmatrix}=0,$$

but $\;|1ab|\,|1cd|+|1bc|\,|1ad|+|1ca|\,|1bd|=0\;$ identically,

therefore $\qquad |1,\ 1/a_1\ a_1|\,|1,\ 1/b_1,\ b_1|=0.$

The conditions for a circle or a rectangular hyperbola are not in general satisfied, as they involve the lengths of the sides of the triangle of reference.

7 lines.

§ 5. The equations of two conics, each related as above to six of 7 straight lines, are

$$\begin{vmatrix} x^2, & xy, & xz, & yz \\ 1, & a_1, & b_1, & a_1b_1 \\ 1, & a_2, & b_2, & a_2b_2 \\ 1, & a_3, & b_3, & a_3b_3 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (14),$$

$$\begin{vmatrix} x^2, & xy, & xw, & yw \\ 1, & a_1, & c_1, & a_1c_1 \\ 1, & a_2, & c_2, & a_2c_2 \\ 1, & a_3, & c_3, & a_3c_3 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (15),$$

where w, c, c_1, c_2, c_3 have values for the seventh line corresponding to the values y, a, a_1, a_2, a_3 for the fifth line.

Eliminating y from these two equations we get $x = 0$ or

$$\begin{vmatrix} x, & \cdot & z, & \cdot \\ 1, & a_1, & b_1, & a_1b_1 \\ 1, & a_2, & b_2, & a_2b_2 \\ 1, & a_3, & b_3, & a_3b_3 \end{vmatrix} \begin{vmatrix} \cdot & x, & \cdot & w \\ 1, & a_1, & c_1, & a_1c_1 \\ 1, & a_2, & c_2, & a_2c_2 \\ 1, & a_3, & c_3, & a_3c_3 \end{vmatrix} = \begin{vmatrix} \cdot & x, & \cdot & z \\ 1, & a_1, & b_1, & a_1b_1 \\ 1, & a_2, & b_2, & a_2b_2 \\ 1, & a_3, & b_3, & a_3b_3 \end{vmatrix} \begin{vmatrix} x, & \cdot & w & \cdot \\ 1, & a_1, & c_1, & a_1c_1 \\ 1, & a_2, & c_2, & a_2c_2 \\ 1, & a_3, & c_3, & a_3c_3 \end{vmatrix}.$$

This equation reduces to the form

$$\begin{vmatrix} x^2, & xz, & xw, & zw \\ 1, & b_1, & c_1, & b_1c_1 \\ 1, & b_2, & c_2, & b_2c_2 \\ 1, & b_3, & c_3, & b_3c_3 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (16),$$

after dividing out the irrelevant factor

$$\begin{vmatrix} 1, & a_1, & a_1^2 \\ 1, & a_2, & a_2^2 \\ 1, & a_3, & a_3^2 \end{vmatrix}.$$

Now there is only one point on both (14) and (15) that lies on $x=0$ or $y=0$, namely, their intersection. Therefore 3 points are common to the conics (14), (15), and (16).

If we write $(7-k)$ to denote the conic derived as above from 6 of 7 straight lines of which the one omitted is no. k, we have $(7-6)$ intersects $(7-7)$ in three points on $(7-k)$ where k is any one of the others.

Therefore the other 5 conics each pass through three of the four intersections of $(7-6)$ and $(7-7)$, and so two of these five must pass through the same 3 points of intersection. Therefore every 4 conics pass through 3 points in common and three of these conics determine the 3 points, and so they all pass through the same 3 points.

In other words the 7 conics derived as above from 6 of 7 given straight lines have 3 points in common.

§ 6. We will now review the whole situation.

If four straight lines are taken, the middle-points of their three diagonals lie on a line, the diameter of the quadrilateral, say A.

If five lines are taken, the 5 diameters A pass through a point the centre of the conic touching the 5 lines, say B.

If six lines are taken, the six points B lie on a conic C.

If seven lines are taken, the 7 conics C pass through 3 points in common.

Arranged in tabular form this appears thus:—

No. of straight lines.	Corresponding property.
4	3 points on a straight line A
5	5 lines A through a point B
6	6 points B on a conic C
7	7 conics C through 3 points D
8	8×3 points D on a cubic E
9	9 cubics E throught 5 points F
10	10×5 points F on a quartic G

where the results 8, 9, 10 have not yet been proved.

8 *lines.*

§ 7. Now let us consider the six conics whose equations are the determinants whose first lines are

$$x^2,\ xy,\ xz,\ yz\ \ldots\ldots\ldots\ldots 1,$$
$$x^2,\ xw,\ xy,\ wy\ \ldots\ldots\ldots\ldots 2,$$
$$x^2,\ xz,\ xw,\ zw\ \ldots\ldots\ldots\ldots 3,$$
$$x^2,\ xw,\ xv,\ vw\ \ldots\ldots\ldots\ldots 4,$$
$$x^2,\ xz,\ xv,\ vz\ \ldots\ldots\ldots\ldots 5,$$
$$x^2,\ xy,\ xv,\ vy\ \ldots\ldots\ldots\ldots 6.$$

Any quartic whose equation is of the form

$$A1.4 + B2.5 + C3.6 = 0\ \ldots\ldots\ldots\ldots (17)$$

will pass through all intersections of the associated conics; for it is satisfied by the intersections of

$$1, 2, 3;\ 1, 5, 6;\ 2, 4, 6;\ \text{or}\ 3, 4, 5:$$

and (17) is the most general form of the equation of a quartic which is so satisfied. Now a cubic can be formed from this by making the coefficient of $vwyz$ zero and dividing by x. This cubic must necessarily pass through all points not on the line $x = 0$, that is, it must pass through the four set of 3 points connected with the groups of conics 1, 2, 3, &c.

The condition that the coefficient of $vwyz$ should be zero in the equation

$$A1.4 + B2.5 + C3.6 = 0$$

is of the form

$$A\,|\,1ab\,|\,|\,1cd\,| + B\,|\,1ca\,|\,|\,1bd\,| + C\,|\,1bc\,|\,|\,1ad\,| = 0.$$

Eliminating A we have

$$B\{|\,1ab\,|\,|\,1cd\,|\,2.5 - |\,1ca\,|\,|\,1bd\,|\,1.4\}$$
$$+\, C\{|\,1ab\,|\,|\,1cd\,|\,3.6 - |\,1bc\,|\,|\,1ad\,|\,1.4\} = 0.$$

Here the ratio $B : C$ is arbitrary, therefore each of the cubics

$$\left.\begin{array}{l}\{|\,1ab\,|\,|\,1cd\,|\,2.5 - |\,1ca\,|\,|\,1bd\,|\,1.4\}/x = 0\\ \{|\,1ab\,|\,|\,1cd\,|\,3.6 - |\,1bc\,|\,|\,1ad\,|\,1.4\}/x = 0\end{array}\right\}\ \ldots (18)$$

passes through the 12 points mentioned above; 9 of these are sufficient to determine the cubic, therefore the cubics are identical, which may be verified.

The four sets of 3 depend on the omission of the lines v, w, y, z from the series considered, and the cubic determined by any three of these omissions passes through the points determined by the fourth omission, but from the geometrical point of view that fourth omission is arbitrary; therefore the cubic (18) passes through the 24 points included in the 8 sets of 3 points.

The cubic (18) can be written in the form

$$\frac{1}{x}\frac{1.4}{|1ab||1cd|}=\frac{1}{x}\frac{2.5}{|1ca||1bd|}=\frac{1}{x}\frac{3.6}{|1bc||1ad|}\quad\text{...(19).}$$

8 *lines (continued).*

§ 8. It seems as if a possible error might exist in assuming that 3 groups of 3 points determined a cubic. The relation between the points might be such as to admit of intersecting cubics, and it might happen that if one of these cubics passed also through one point of another group that necessitated its passing through all the 3 points of the group. This, if true, would give rise to 70 cubics each passing through the points of 4 of the eight groups, which seems more unlikely than one passing through 24 related points. The form of equation however that we have obtained being only symmetrically related to 4 of the given lines does not lend itself to the elucidation of this point. I have therefore proceeded to work out the equation of the cubic of 12 points for the 8 lines 2, 3, 4, 5, 6, 7, 8, 9.

The centre of the conic touching lines 2, 3, 4, 5, 6 is

$$x=\frac{y}{a}=\frac{z}{b},$$

and the centres of the conics touching the lines 2, 3, 4, 5, 7 and 2, 3, 4, 6, 7,

$$x=\frac{y}{a}=\frac{w}{c},$$

$$x=\frac{z}{b}=\frac{w}{c}.$$

These three points lie on the conic

$$A\left(x-\frac{y}{b}\right)\left(x-\frac{w}{c}\right)+B\left(x-\frac{w}{c}\right)\left(x-\frac{y}{a}\right)+C\left(x-\frac{y}{a}\right)\left(x-\frac{z}{b}\right)=0,$$

or $$A\left(x-\frac{y}{a}\right)^{-1}+B\left(x-\frac{z}{b}\right)^{-1}+C\left(x+\frac{w}{c}\right)^{-1}=0.$$

The centre of the conic touching 3, 4, 5, 6, 7 is the intersection of the diameters of 3, 4, 5, 6 and 3, 4, 6, 7.

The diameter of 3, 4, 5, 6 passes through the centres of the conics touching 2, 3, 4, 5, 6 and 1, 3, 4, 5, 6; that is, through the points

$$x = \frac{y}{a} = \frac{z}{b} \text{ and } x = \frac{y}{a_1} = \frac{z}{b_1}.$$

Therefore its equation is

$$\begin{vmatrix} x, & y, & z \\ 1, & a, & b \\ 1, & a_1, & b_1 \end{vmatrix} = 0, \text{ or } \frac{(ax-y)}{a-a_1} = \frac{bx-z}{b-b_1}.$$

Similarly the equation of 3, 4, 6, 7 is

$$\begin{vmatrix} x, & z, & w \\ 1, & b, & c \\ 1, & b_1, & c_1 \end{vmatrix} = 0, \text{ or } \frac{bx-z}{b-b_1} = \frac{cz-w}{c-c_1}.$$

The intersection of these two lines is

$$\frac{ax-y}{a-a_1} = \frac{bx-z}{b-b_1} = \frac{cx-w}{c-c_1}.$$

Similarly the centre of the conic 2, 4, 5, 6, 7 is

$$\frac{ax-y}{a-a_2} = \frac{bx-z}{b-b_2} = \frac{cx-w}{c-c_2}.$$

Therefore the equation of the conic containing the centres of the conics 2, 3, 4, 5, 6, 7 is

$$\begin{vmatrix} (ax-y)^{-1}, & (bx-z)^{-1}, & (cx-w)^{-1} \\ (a-a_1)^{-1}, & (b-b_1)^{-1}, & (c-c_1)^{-1} \\ (a-a_2)^{-1}, & (b-b_2)^{-1}, & (c-c_2)^{-1} \end{vmatrix} = 0.$$

As the conic also passes through the centre of 2, 3, 5, 6, 7, it is necessary that the condition below should be satisfied, and this is easily verified.

$$\begin{vmatrix} (a-a_1)^{-1}, & (b-b_1)^{-1}, & (c-c_1)^{-1} \\ (a-a_2)^{-1}, & (b-b_2)^{-1}, & (c-c_2)^{-1} \\ (a-a_3)^{-1}, & (b-b_3)^{-1}, & (c-c_3)^{-1} \end{vmatrix} = 0.$$

The equation of 2, 3, 4, 5, 8, 9 is, with a similar notation

$$\begin{vmatrix} 1/(ax-y), & 1/(dx-v), & 1/(ex-u) \\ 1/(a-a_1), & 1/(d-d_1), & 1/(e-e_1) \\ 1/(a-a_2), & 1/(d-d_2), & 1/(e-e_2) \end{vmatrix},$$

The equations of 2, 3, 4, 5, 6, 8 and 2, 3, 4, 5, 7, 9 are similar.

The equation of the cubic passing through all the 3-point intersections of the conics 234567, 234589 with conics 234568, 234579 is

$$\frac{\begin{vmatrix} (ax-y)^{-1}, & (bx-z)^{-1}, & (cx-w)^{-1} \\ (a-a_1)^{-1}, & (b-b_1)^{-1}, & (c-c_1)^{-1} \\ (a-a_2)^{-1}, & (b-b_2)^{-1}, & (c-c_2)^{-1} \end{vmatrix} \begin{vmatrix} (ax-y)^{-1}, & (dx-v)^{-1}, & (ex-u)^{-1} \\ (a-a_1)^{-1}, & (d-d_1)^{-1}, & (e-e_1)^{-1} \\ (a-a_2)^{-1}, & (d-d_2)^{-1}, & (e-e_2)^{-1} \end{vmatrix}}{\begin{vmatrix} (b-b_1)^{-1}, & (c-c_1)^{-1} \\ (b-b_2)^{-1}, & (c-c_2)^{-1} \end{vmatrix} \begin{vmatrix} (d-d_1)^{-1}, & (e-e_1)^{-1} \\ (d-d_2)^{-1}, & (e-e_2)^{-1} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} (ax-y)^{-1}, & (bx-z)^{-1}, & (dx-v)^{-1} \\ (a-a_1)^{-1}, & (b-c_1)^{-1}, & (d-d_1)^{-1} \\ (a-a_2)^{-1}, & (b-b_2)^{-1}, & (d-d_2)^{-1} \end{vmatrix} \begin{vmatrix} (ax-y)^{-1}, & (cx-w)^{-1}, & (ex-u)^{-1} \\ (a-a_1)^{-1}, & (c-c_1)^{-1}, & (e-e_1)^{-1} \\ (a-a_2)^{-1}, & (c-c_2)^{-1}, & (e-e_2)^{-1} \end{vmatrix}}{\begin{vmatrix} (b-b_1)^{-1}, & (d-d_1)^{-1} \\ (b-b_2)^{-1}, & (d-d_2)^{-1} \end{vmatrix} \begin{vmatrix} (c-c_1)^{-1}, & (e-e_1)^{-1} \\ (c-c_2)^{-1}, & (e-e_2)^{-1} \end{vmatrix}}.$$

Writing 5, 6, 7, &c., for the terms in the first line of the determinants in the numerators on both sides of this equation, and (bc), (ca), (ab), &c., for the minors of those terms, the equation can be written

$$(bd)(ce)\{5(bc)+6(ca)+7(ab)\}\{5(de)+8(ea)+9(ad)\}$$
$$=(bc)(de)\{5(bd)+6(da)+8(ab)\}\{5(ce)+7(ea)+9(ac)\},$$

or

$$\begin{aligned} & 5.8\{(bc)(ea)(bd)(ce)-(bc)(de)(ab)(ce)\} \\ +\,& 5.9\{(bc)(ad)(bd)(ce)-(bc)(de)(bd)(ac)\} \\ +\,& 5.6\{(ca)(de)(bd)(ce)-(bc)(de)(ce)(da)\} \\ +\,& 5.7\{(ab)(de)(bd)(ce)-(bc)(de)(bd)(ea)\} \\ +\,& 6.8\,(ca)(ea)(bd)(ce) \quad -6.7\,(bc)(de)(da)(ea) \\ +\,& 6.9\{(ca)(ad)(bd)(ce)-(bc)(de)(da)(ac)\} \\ +\,& 7.8\{(ab)(ea)(bd)(ce)-(bc)(de)(ab)(ea)\} \\ +\,& 7.9\,(ab)(ad)(bd)(ce) \quad -8.9\,(bc)(de)(ab)(ac), \end{aligned}$$

or by virtue of the relations

$$(ea)(bd)+(ed)(ab)+(eb)(da)=0,$$

$$\left.\begin{aligned} &5.6\,(cd)(de)(ec)(ab)\\ +&5.7\,(bd)(de)(eb)(ca)\\ +&5.8\,(bc)(ce)(eb)(ad)\\ +&5.9\,(bc)(db)(cd)(ea)\\ +&6.7\,(ad)(de)(ea)(bc)\\ +&6.8\,(ac)(ce)(ea)(db)\\ +&6.9\,(ac)(cd)(da)(be)\\ +&7.8\,(ab)(be)(ea)(cd)\\ +&7.9\,(ab)(bd)(da)(ec)\\ +&8.9\,(ab)(bc)(ca)(de) \end{aligned}\right\}=0.$$

A form that is found to be symmetrical by a cyclical interchange, and an interchange of two letters either everywhere changes the sign or everywhere leaves it unaltered.

The symmetry of the equation shows that the cubic has the same relation to the four groups of conics in which 2346, 2347, 2348, or 2349 are constant elements and therefore passes through 5 sets of 3 points. Hence there is only one cubic and it passes through all the 24 points.

9 lines.

§ 9. Let $\quad Ax^2+Bxy+Cxz+yz=1/|\,1ab\,|$

and also $\qquad\qquad\qquad\qquad =(yz),$

and let $\quad A_{mn}x^2+B_{mn}xy_m+C_{mn}xz_n+y_mz_n=(y_mz_n)$

be a similar expression with other variables y_m, z_n.

With this notation one cubic can be written

$$(y_1z_1)(y_2z_2)/x=(y_1z_2)(y_2z_1)/x \text{..........(20)},$$

and a second cubic with one of the 8 lines altered can be written

$$(y_1z_1)(y_3z_2)/x=(y_1z_2)(y_3z_1)/x \text{..........(21)}.$$

These cubics meet one another where

$$(y_1z_1)(y_1z_2)=0 \text{..........(22)},$$

or where $\quad (y_2z_2)(y_3z_1)/x=(y_2z_1)(y_3z_2)/x \text{..........(23)}.$

Now (y_1z_1), (y_1z_2) each only meets both cubics (20) and (21) where the other does, or at points on both (y_2z_1), (y_3z_1) or on both (y_2z_2), (y_3x_2), *i.e.* on (23), so that four of the points common to the two cubics lie on both these conics and the remaining five points of their intersection lie on the cubic (23), which omits the line y_1, and by the same argument that has been used above for the conics; these 5 points lie on all the cubics connected with the 9 lines.

Lemma:—

§ 10.

$$\begin{vmatrix} (y_1z_1), & (y_2z_1), & (y_3z_1), & \dots, & (y_nz_1) \\ (y_1z_2), & (y_2z_2), & (y_3z_2), & \dots, & (y_nz_2) \\ (y_1z_3), & (y_2z_3), & (y_3z_3), & \dots, & (y_nz_3) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (y_1z_n), & (y_2z_n), & (y_3z_n), & \dots, & (y_nz_n) \end{vmatrix} \qquad \dots\dots\dots\dots(24)$$

is divisible by x^{n-1}.

For, multiplying every column except the first by y_1 and subtracting from the r^{th} column y_r times the first column, the term in the s^{th} row becomes

$$A_{rs}x^2y_1 + B_{rs}xy_1y_r + C_{rs}xy_1z_s + y_1y_rz_s$$
$$- A_{1s}x^2y_r - B_{1s}xy_1y_r - C_{1s}xy_rz_s - y_1y_rz_s,$$

or $\quad x\{x(A_{rs}y_1 - A_{1s}y_r) + (B_{rs} - B_{1s})y_1y_r + (C_{rs}y_1 - C_{1s}y_r)z_s\}.$

Therefore each column except the first as so altered will divide through by x, but the determinant will then have been multiplied by y_1^{n-1}, which does not contain the factor x. So the original determinant contains the factor x^{n-1}.

It may also be observed that no squares of any of the variables y, z occur in the expanded determinant and that every product of these variables does occur.

10 *lines.*

§ 11. The cubics

$$\begin{vmatrix} (y_1\ z_1), & (y_2\ z_1) \\ (y_1\ z_2), & (y_2\ z_2) \end{vmatrix} \Big/ x = 0, \quad \begin{vmatrix} (y_1\ z_1), & (y_3\ z_1) \\ (y_1\ z_2), & (y_3\ z_2) \end{vmatrix} \Big/ x = 0,$$

meet the cubics

$$\begin{vmatrix} (y_1\ z_1), & (y_2\ z_1) \\ (y_1\ z_3), & (y_2\ z_3) \end{vmatrix} \Big/ x = 0, \quad \begin{vmatrix} (y_1\ z_1), & (y_3\ z_1) \\ (y_1\ z_2), & (y_3\ z_2) \end{vmatrix} \Big/ x = 0,$$

where

$$\begin{Vmatrix} \begin{vmatrix} (y_1\ z_1), & (y_2\ z_1) \\ (y_1\ z_2), & (y_2\ z_2) \end{vmatrix} & \begin{vmatrix} (y_1\ z_1), & (y_3\ z_1) \\ (y_1\ z_3), & (y_3\ z_3) \end{vmatrix} \\ \begin{vmatrix} (y_1\ z_1), & (y_2\ z_1) \\ (y_1\ z_3), & (y_2\ z_3) \end{vmatrix} & \begin{vmatrix} (y_1\ z_1), & (y_3\ z_1) \\ (y_1\ z_2), & (y_3\ z_2) \end{vmatrix} \end{Vmatrix} \Big/ x^2 = 0,$$

that is where $(y_1\ z_1) = 0$ and where

$$\begin{vmatrix} (y_1\ z_1), & (y_2\ z_1), & (y_3\ z_1) \\ (y_1\ z_2), & (y_2\ z_2), & (y_3\ z_2) \\ (y_1\ z_3), & (y_2\ z_3), & (y_3\ z_3) \end{vmatrix} \Big/ x^2 = 0 \ldots\ldots (25).$$

That is the 20 points (4 sets of 5) lie on the quartic (25), and as 15 of these are sufficient to determine the quartic this curve must also pass through the other sets of 5 points—a result that could also be arrived at by considering the symmetry of the equation (25).

General case.

§ 12. The reasoning used in the case of the 9 lines applies with slight modifications to the case of any odd number and the reasoning in the case of 10 lines applies similarly to the case of any even number by the help of the appeal to symmetry.

Therefore with $2n$ lines we have an $(n-1)^{\text{ic}} X$;

with $2n+1$ lines we have $2n+1$ $(n-1)^{\text{ics}} X$

and these all pass through $2n-3$ points Y;

with $2n+2$ lines we have $(2n+2)(2n-3)$ points Y;

and these all lie on an $n^{\text{ic}} Z$.

ON VARIOUS EXPRESSIONS FOR h, THE NUMBER OF PROPERLY PRIMITIVE CLASSES FOR A NEGATIVE DETERMINANT NOT CONTAINING A SQUARE FACTOR.

(Fifth Paper.)

By *H. Holden.*

1. The present paper is an attempt to deduce, for $p=4n+1$, general relations connecting the sums of the quadratic characters of the integers contained between certain limits, similar to those obtained for $p=4n+3$, in the second paper of this series.

2. Writing H for $\dfrac{h}{2-(2/p)}$, it has been proved that, when $p=4n+3$,

$$\Sigma\frac{1}{1-r^{\alpha}}-\Sigma\frac{1}{1-r^{\beta}}=i\sqrt{p}\,.\,H,$$

and
$$\Sigma\frac{1}{1+r^{\alpha}}-\Sigma\frac{1}{1+r^{\beta}}=(2/p)\,.\,i\sqrt{p}\,.\,h;$$

whence
$$\Sigma\frac{1}{1-r^{2\alpha}}-\Sigma\frac{1}{1-r^{2\beta}}=(2/p)\,.\,i\sqrt{p}\,.\,H,$$

and
$$\Sigma\frac{1-r^{\alpha}}{1-r^{2\alpha}}-\Sigma\frac{1-r^{\beta}}{1-r^{2\beta}}=(2/p)\,.\,i\sqrt{p}\,.\,h;$$

therefore
$$\Sigma\frac{r^{\alpha}}{1-r^{2\alpha}}-\Sigma\frac{r^{\beta}}{1-r^{2\beta}}=i\sqrt{p}\,.\,H\{1-(2/p)\}.$$

3. Writing $\Sigma\dfrac{r^{\alpha}}{1-r^{2\alpha}}-\Sigma\dfrac{r^{\beta}}{1-r^{2\beta}}$ as $\Sigma\,(\mu/p)\dfrac{r^{\mu}}{1-r^{2\mu}}$, where the limits of μ are 1 and $p-1$, we have

$$\Sigma\,(\mu/p)\frac{r^{\mu}}{1-r^{2\mu}}=(q/p)\,\Sigma\,(\mu/p)\frac{r^{q\mu}}{1-r^{2q\mu}},$$

and also
$$=\Sigma\,(\mu/p)\frac{r^{\mu}+r^{3\mu}+r^{5\mu}+\ldots+r^{(2q-1)\mu}}{1-r^{2q\mu}}.$$

Firstly, let q be even and prime to p.

Then if
$$\begin{aligned} p + l_1 &\equiv q \pmod{2q}, \\ 3p + l_2 &\equiv q \pmod{2q}, \\ &\vdots \\ (2q-1)\,p + l_q &\equiv q \pmod{2q}, \end{aligned}$$
it is clear that $l_1, l_2, \ldots, l_q$, are the odd integers less than $2q$.

Now

$$q\,\Sigma\,(\mu/p)\,\frac{r^{q\mu}}{1-r^{2q\mu}}$$

$$= \Sigma\,(\mu/p) \left\{ \begin{array}{ll} r^{q\mu} + r^{3q\mu} + \ldots + r^{(p+l_1-2q)\mu} & + \dfrac{r^{(p+l_1)\mu}}{1-r^{2q\mu}} \\ + r^{q\mu} + r^{3q\mu} + \ldots + r^{(3p+l_2-2q)\mu} & + \dfrac{r^{(3p+l_2)\mu}}{1-r^{2q\mu}} \\ \vdots \quad \vdots \quad\quad \vdots & \quad\quad \vdots \\ + r^{q\mu} + r^{3q\mu} + \ldots + r^{\{(2q-1)p+l_q-2q\}\mu} & + \dfrac{r^{\{(2q-1)p+l_q\}\mu}}{1-r^{2q\mu}} \end{array} \right\},$$

and therefore

$$q.(q/p).\{1-(2/p)\}H = (q/p)\{\Sigma_0^{p/q}(n/p) + \Sigma_0^{3p/q}(n/p) + \ldots + \Sigma_0^{\{(2q-1)p\}/q}(n/p)\}$$
$$+ \{1-(2/p)\,H\},$$

where n is an odd integer.

Dividing by (q/p), we get

$$\{q-(q/p)\}\{1-(2/p)\}H = \Sigma_0^{p/q}(n/p) + \Sigma_0^{3p/q}(n/p) + \ldots + \Sigma_0^{\{(2q-1)p\}/q}(n/p)$$
$$= q\,\Sigma_0^{p/q}(n/p) + (q-2)\,\Sigma_{p/q}^{3p/q}(n/p) + (q-4)\,\Sigma_{3p/q}^{5p/q}(n/p) + \ldots,$$

he series terminating with the last positive coefficient.

Secondly, let q be odd and prime to p.

Then if
$$\begin{aligned} 0.p + q &\equiv q \pmod{2q}, \\ 2p + m_1 &\equiv q \pmod{2q}, \\ &\vdots \\ (2q-2)\,p + m_{q-1} &\equiv q \pmod{2q}, \end{aligned}$$
it is clear that $q, m_1, m_2, \ldots, m_{q-1}$, are the odd integers less than $2q$, and that the result is much the same as before, except that on the right hand side we have $q-1$ terms only.

Therefore, if q be odd and prime to p,

$$\{q-(q/p)\}\{1-(2/p)\}H=\Sigma_0^{2p/q}(n/p)+\Sigma_0^{4p/q}(n/p)+\ldots+\Sigma_0^{\{(2q-2)p\}/q}(n/p)$$

$$=(q-1)\,\Sigma_0^{2p/q}(n/p)+(q-3)\,\Sigma_{2p/q}^{4p/q}(n/p)+\ldots,$$

where, as before, n is an odd integer, and the series terminates with the last positive coefficient.

4. Suppose now that $p=4n+1$.
Then, since

$$\Sigma\,(\mu/p)\,\frac{r^{\mu}}{1-r^{4\mu}}=\Sigma\,(\mu/p)\,\frac{r^{-\mu}}{1-r^{-4\mu}}=-\,\Sigma\,(\mu/p)\,\frac{r^{3\mu}}{1-r^{4\mu}},$$

and $\Sigma\,(\mu/p)\,\dfrac{r^{\mu}}{1-r^{4\mu}}-\Sigma\,(\mu/p)\,\dfrac{r^{3\mu}}{1-r^{4\mu}}=\Sigma\,(\mu/p)\,\dfrac{r^{\mu}-r^{3\mu}}{1-r^{4\mu}}$

$$=\Sigma\,(\mu/p)\,\frac{r^{\mu}}{1+r^{2\mu}}$$

$$=\sqrt{p}\,.\,h,$$

we have $\Sigma\,(\mu/p)\,\dfrac{r^{\mu}}{1-r^{4\mu}}=-\,\Sigma\,(\mu/p)\,\dfrac{r^{3\mu}}{1-r^{4\mu}}=\sqrt{p}\,.\,\tfrac{1}{2}h,$

and also $\Sigma\,(\mu/p)\,\dfrac{r^{\mu}}{1-r^{2\mu}}=0$, the limits of μ being 1 and $p-1$, in each case.

Then from $\Sigma\,(\mu/p)\,\dfrac{r^{\mu}}{1-r^{4\mu}}=\sqrt{p}\,.\,\tfrac{1}{2}h$, by a method similar to that used in the preceding section, the following relations could be deduced. It is necessary to distinguish 4 cases, according as $q\equiv 0$, 1, 2 or 3 (mod. 4).

For $q=4n+1$,

$$\{q-(q/p)\}\,\tfrac{1}{2}h=\Sigma_0^{4p/q}\,(n/p)+\Sigma_0^{8p/q}\,(n/p)+\ldots+\Sigma_0^{\{(4q-4)p\}/q}\,(n/p)$$

$$=(q-1)\,\Sigma_0^{4p/q}\,(n/p)+(q-2)\,\Sigma_{4p/q}^{8p/q}\,(n/p)+\ldots.$$

For $q=4n+2$,

$$\{q-(q/p)\}\,\tfrac{1}{2}h=\Sigma_0^{p/q}\,(n/p)+\Sigma_0^{5p/q}\,(n/p)+\ldots+\Sigma_0^{\{(4q-3)p\}/q}\,(n/p)$$

$$=q\,\Sigma_0^{p/q}\,(n/p)+(q-1)\,\Sigma_{p/q}^{5p/q}\,(n/p)+\ldots.$$

For $q = 4n + 3$,

$$\{q-(q/p)\}\tfrac{1}{2}h = \Sigma_0^{2p/q}(n/p) + \Sigma_0^{6p/q}(n/p) + \ldots + \Sigma_0^{\{(4q-2)p\}/q}(n/p)$$

$$= q\Sigma_0^{2p/q}(n/p) + (q-1)\,\Sigma_{2p/q}^{6p/q}(n/p) + \ldots.$$

For $q = 4n$,

$$\{q-(q/p)\}\tfrac{1}{2}h = \Sigma_0^{3p/q}(n/p) + \Sigma_0^{7p/q}(n/p) + \ldots + \Sigma_0^{\{(4q-1)p\}/q}(n/p)$$

$$= q\Sigma_\theta^{3p/q}(n/p) + (q-1)\,\Sigma_{3p/q}^{7p/q}(n/p) + \ldots.$$

In each case n is restricted to integers of the form $4m+1$, and the series ends with the last positive coefficient.

As these equations do not apparently yield what was wanted, a different mode of procedure was adopted.

5. Since

$$\Sigma(\mu/p)\frac{1}{1-r^{q\mu}} = \Sigma(\mu/p)\frac{1}{1-r^{-q\mu}} = -\Sigma(\mu/p)\frac{r^{q\mu}}{1-r^{q\mu}},$$

and $\Sigma(\mu/p)\dfrac{1}{1-r^{q\mu}} - \Sigma(\mu/p)\dfrac{r^{q\mu}}{1-r^{q\mu}} = \Sigma(\mu/p) = 0,$

ve have $\Sigma(\mu/p)\dfrac{1}{1-r^{q\mu}} = 0$, and this relation, combined with thers previously obtained, and similar types, give series which nay be expanded without leaving any fractional remainders.

To illustrate the method adopted, take

$$\Sigma(\mu/p)\frac{r^{\mu}}{1-r^{2\mu}} = 0.$$

Combining this with

$$\Sigma(\mu/p)\frac{1}{1-r^{2\mu}} = 0,$$

e have
$$\Sigma(\mu/p)\frac{1-r^{\mu}}{1-r^{2\mu}} = 0,$$

r
$$\Sigma(\mu/p)\frac{1-r^{(p+1)\mu}}{1-r^{2\mu}} = 0,$$

r
$$\Sigma(\mu/p)(1+r^{2\mu}+\ldots+r^{(p-1)\mu}) = 0;$$

or, dividing by $(2/p)\sqrt{p}$, we get

$$\Sigma_0^{\frac{1}{2}(p-1)}(n/p)=0,$$

where n is any positive integer.

Similarly from

$$\Sigma\,(\mu/p)\,\frac{r^{3\mu}}{1-r^{4\mu}}=-\sqrt{p}\,.\,\tfrac{1}{2}h,$$

we have
$$\Sigma\,(\mu/p)\,\frac{1-r^{(p+3)\mu}}{1-r^{4\mu}}=\sqrt{p}\,.\,\tfrac{1}{2}h,$$

or
$$\Sigma_0^{(p-1)/4}(n/p)=\tfrac{1}{2}h.$$

These results are, of course, already known.

As another example of the method, take

$$\sqrt{p}\,.\,\tfrac{1}{2}h=\Sigma\,(\mu/p)\,\frac{r^{\mu}}{1-r^{4\mu}}=\Sigma\,(\mu/p)\,\frac{r^{\mu}+r^{5\mu}+\ldots+r^{(4q-3)\mu}}{1-r^{4q\mu}},$$

which gives

$$-(q/p)\,\tfrac{1}{2}h=\Sigma_0^{3p/4q}(n/p)+\Sigma_0^{7p/4q}(n/p)+\ldots+\Sigma_0^{\{(4q-1)p\}/4q}(n/p),$$

where n is any positive integer.

Similarly $-\sqrt{p}\,.\,\tfrac{1}{2}h=\Sigma\,(\mu/p)\,\dfrac{r^{3\mu}}{1-r^{4\mu}}$ gives

$$(q/p)\,\tfrac{1}{2}h=\Sigma_0^{p/4q}(n/p)+\Sigma_0^{5p/4q}(n/p)+\ldots+\Sigma_0^{\{(4q-3)p\}/4q}(n/p).$$

Subtracting these two results,

$$(q/p)\,h=\Sigma_0^{p/4q}(n/p)+\Sigma_{3p/4q}^{5p/4q}(n/p)+\ldots+\Sigma_{\{(4q-1)p\}/4q}^{p}(n/p).$$

Writing a_r for $\Sigma_{\{(r-1)p\}/4q}^{rp/4q}(n/p)$, we have

$$(q/p)\,h=a_1+a_4+a_5+a_8+a_9+\ldots+a_{4q}$$
$$=2\,(a_1+a_4+a_5 \text{ up to } a_{2q}).$$

Thus if $q=2$, and $4q=8$, we have the following relation between the 8th intervals

$$a_1+a_4=(2/p)\,\tfrac{1}{2}h.$$

If $q=3$ and $4q=12$, we have the following relation between the 12th intervals

$$(3/p)\,h=2\,(a_1+a_4+a_5)$$
$$=2\,(a_1-\tfrac{1}{2}h-a_6),$$

or

$$a_1-a_6=[\tfrac{1}{2}\{1+(3/p)\}]\,.\,h.$$

6. Let $q=6$, and write a_r for $\Sigma_{\frac{1}{6}\{(r-1)p\}}^{\frac{1}{6}(rp)}(n/p)$, where n is any positive integer, we have

$$\Sigma\,(\mu/p)\,\frac{r^{\mu}}{1-r^{3\mu}}=(2/p)\,\Sigma\,(\mu/p)\,\frac{r^{2\mu}}{1-r^{6\mu}}=\Sigma\,(\mu/p)\,\frac{r^{\mu}+r^{4\mu}}{1-r^{6\mu}}\,.$$

Therefore

$$(2/p)\,\Sigma\,(\mu/p)\,\frac{1-r^{2\mu}}{1-r^{6\mu}}=\Sigma\,(\mu/p)\,\frac{1-r^{\mu}}{1-r^{6\mu}}+\Sigma\,(\mu/p)\,\frac{1-r^{4\mu}}{1-r^{6\mu}}\,.$$

For $p=24n+1$, $(2/p)=1$, and

$$\Sigma\,(\mu/p)\,\frac{1-r^{(4p+3)\mu}}{1+r^{6\mu}}=\Sigma\,(\mu/p)\,\frac{1-r^{(5p+1)\mu}}{1-r^{6\mu}}+\Sigma\,(\mu/p)\,\frac{1-r^{(2p+4)\mu}}{1-r^{6\mu}}\,,$$

or

$$\Sigma_0^{\frac{1}{6}(4p)}(n/p)=\Sigma_0^{\frac{1}{6}(5p)}(n/p)+\Sigma_0^{\frac{1}{6}(2p)}(n/p),$$

or

$$-a_5-a_6=-a_6+a_1+a_2,$$

or

$$a_1+2a_2=0,$$

or

$$a_2=a_3.$$

7. The same result would be got if $p=24n+17$.
Therefore, for $p=8n+1$, if prime to 3,

$$a_2=a_3=-\tfrac{1}{2}a_1.$$

Proceeding similarly, if $p=8n+5$ and prime to 3, we should get

$$a_1=0,$$
$$a_2=-a_3.$$

No connection with h is got, because q is not of the form $4n$.

8. For $q=8$ and $a_r=\Sigma_{\frac{1}{8}\{(r-1)p\}}^{\frac{1}{8}(rp)}(n/p)$, we have already found that $a_1+a_4=(2/p)\,\tfrac{1}{2}h$.

Combining this with the known results

$$a_1 + a_2 = \tfrac{1}{2}h,$$

and
$$a_1 + a_2 + a_3 + a_4 = 0,$$

we have, for $p = 8n + 1$,

$$a_1 + a_4 = \tfrac{1}{2}h,$$

$$a_2 = a_4,$$

$$a_1 - a_3 = h;$$

and, for $p = 8n + 5$,

$$a_1 + a_4 = -\tfrac{1}{2}h,$$

$$a_1 = a_3,$$

$$a_2 - a_4 = h.$$

9. If $q = 12$, we get three distinct relations by starting with sums of expressions of the types $\frac{r^{\mu}}{1 - r^{4\mu}}$, $\frac{r^{\mu}}{1 - r^{6\mu}}$, and $\frac{r^{2\mu}}{1 - r^{6\mu}}$, and treating them as in the 6th section. No fresh relation is got by taking an expression for which the power of r in the numerator is not less than one-half the power of r in the denominator.

For $p = 24n + 1$, the three relations are

$$a_2 = a_6,$$

$$a_3 = a_5,$$

$$a_1 - a_6 = h,$$

which, combined with

$$a_1 + a_2 + a_3 = \tfrac{1}{2}h,$$

$$a_4 + a_5 + a_6 = -\tfrac{1}{2}h,$$

yield
$$a_2 = a_4 = a_6,$$

$$a_3 = a_5,$$

$$a_1 - a_2 = h.$$

$$a_3 + 2a_4 = -\tfrac{1}{2}h.$$

For $p=24n+5$, the relations reduce to

$$a_1=a_2=a_6=0,$$
$$a_3=\tfrac{1}{2}h,$$
$$a_4+a_5=-\tfrac{1}{2}h.$$

For $p=24n+13$, the relations reduce to

$$a_1=-a_2=a_3=-a_6=\tfrac{1}{2}h,$$
$$a_4=-a_5.$$

For $p=24n+17$, the relations reduce to

$$a_1=a_2=a_6=-\tfrac{1}{2}a_5,$$
$$a_1-a_4=\tfrac{1}{2}h,$$
$$a_3+2a_4=-\tfrac{1}{2}h.$$

10. The following examples show the kind of result got for $q=18$.

For $p=72n+1$,

$$a_2=a_5=a_7=a_9,$$
$$a_3=a_6,$$
$$a_2+a_8=a_3+a_4,$$
$$a_1+3a_2+3a_3+2a_4=0.$$

For $p=72n+5$,

$$a_5=a_9=0,$$
$$a_1=a_3=-\tfrac{1}{2}(a_2)=a_4+a_8=-a_6-a_7.$$

11. Relations between the intervals (not lower than the 8th) and values of h for even negative determinants may also be got.

Thus if $-D=2p=8n+2$, we have

$$\sqrt{p}\,.\,\tfrac{1}{2}h=\Sigma(\mu/p)\frac{r^{\mu}+r^{-\mu}}{r^{2\mu}+r^{-2\mu}}=\Sigma(\mu/p)\frac{r^{\mu}+r^{3\mu}}{1+r^{4\mu}}=\Sigma(\mu/p)\frac{r^{\mu}+r^{3\mu}-r^{5\mu}-r^{7\mu}}{1-r^{8\mu}}.$$

But

$$\Sigma(\mu/p)\frac{r^{\mu}+r^{3\mu}}{1-r^{8\mu}}=\Sigma(\mu/p)\frac{r^{-\mu}+r^{-3\mu}}{1-r^{-8\mu}}=-\Sigma(\mu/p)\frac{r^{7\mu}+r^{5\mu}}{1-r^{8\mu}},$$

therefore $\Sigma(\mu/p)\dfrac{r^{\mu}+r^{3\mu}}{1-r^{8\mu}}=\sqrt{p}\,.\,\tfrac{1}{2}h$, which may be treated as above.

If $-D=2p=8n+6$, we have

$$-i\sqrt{p}\,.\,\tfrac{1}{2}h=\Sigma(\mu/p)\frac{r^{\mu}-r^{-\mu}}{r^{2\mu}+r^{-2\mu}}=\Sigma(\mu/p)\frac{r^{3\mu}-r^{\mu}}{1+r^{4\mu}}=\Sigma(\mu/p)\frac{r^{3\mu}-r^{\mu}-r^{7\mu}+r^{5\mu}}{1-r^{8\mu}}.$$

But

$$\Sigma(\mu/p)\frac{r^{3\mu}-r^{\mu}}{1-r^{8\mu}}=-\Sigma(\mu/p)\frac{r^{-3\mu}-r^{-\mu}}{1-r^{-8\mu}}=+\Sigma(\mu/p)\frac{r^{5\mu}-r^{7\mu}}{1-r^{8\mu}},$$

therefore

$$\Sigma(\mu/p)\frac{r^{\mu}-r^{3\mu}}{1-r^{8\mu}}=i\sqrt{p}\,.\,\tfrac{1}{2}h.$$

Otherwise, both for $2p=8n+2$, and $8n+6$, we have

$$(\eta_0-\eta_1)\,\tfrac{1}{2}h=\Sigma(\mu/p)\frac{r^{\mu}-r^{5\mu}}{1-r^{8\mu}}.$$

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CERTAIN FUNDAMENTAL QUANTITIES IN THE THEORY OF TORTUOUS CURVES.

By *F. B. Pidduck*, Scholar of Exeter College, Oxford.

1. The object of the present note is to call attention to the importance of a certain set of quantities connected with a curve in space, and to find expressions for curvature, torsion, and similar measures in terms of them. If x, y, z, the coordinates of a point on a curve, are expressed in terms of a parameter t, and suffixes denote differentiation with respect to t, these fundamental quantities are

$$a_{rs}\equiv x_r x_s+y_r y_s+z_r z_s\equiv\Sigma x_r x_s.$$

Their importance lies in the fact that they are unaltered by changes of rectangular axes. For they are clearly

unchanged by a displacement of the origin, and the orthogonal substitution

$$x = l_1x' + l_2y' + l_3z',$$
$$y = m_1x' + m_2y' + m_3z',$$
$$z = n_1x' + n_2y' + n_3z'$$

changes $\Sigma x_r x_s$ into

$$\Sigma\,(l_1x_r' + l_2y_r' + l_3z_r')\,(l_1x_s' + l_2y_s' + l_3z_s'),$$

or into $\Sigma x_r' x_s'$ by virtue of the orthogonal relations connecting l_1, m_1, &c.

The greater of the numbers r, s determines the *order* of the quantities. With a slight change of notation we may enumerate those which will be of immediate use, namely,

$$a \equiv \Sigma x_1^2, \qquad b \equiv \Sigma x_2^2, \qquad c \equiv \Sigma x_3^2,$$
$$f \equiv \Sigma x_2 x_3, \qquad g \equiv \Sigma x_3 x_1, \qquad h \equiv \Sigma x_1 x_2.$$

In what follows we shall make use of the following identities, which are capable of immediate verification, but are also particular cases of a general theorem on minors of product determinants*:—

$$\left.\begin{aligned} \Sigma\,(y_2z_3 - y_3z_2)^2 &= A \\ \Sigma\,(y_3z_1 - y_1z_3)^2 &= B \\ \Sigma\,(y_1z_2 - y_2z_1)^2 &= C \\ \Sigma\,(y_3z_1 - y_1z_3)\,(y_1z_2 - y_2z_1) &= F \\ \Sigma\,(y_1z_2 - y_2z_1)\,(y_2z_3 - y_3z_2) &= G \\ \Sigma\,(y_2z_3 - y_3z_2)\,(y_3z_1 - y_1z_3) &= H \end{aligned}\right\} \text{............(1)},$$

where $\quad A = bc - f^2, \;\; F = gh - af,\;$ &c.

The results of differentiating some of these quantities are also written down for convenience:—

$$\left.\begin{aligned} \frac{da}{dt} = 2h, \quad \frac{db}{dt} = 2f, \quad \frac{dh}{dt} = b + g \\ \frac{dC}{dt} = -2F \end{aligned}\right\} \text{..........(2)}.$$

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* Scott's *Determinants*, p. 53.

2. If r is the distance between $(\xi\eta\zeta)$ and (xyz), we have

$$(\xi-x)^2+(\eta-y)^2+(\zeta-z)^2=r^2 \quad\ldots\ldots\ldots\ldots(3).$$

Differentiating with respect to t, keeping r constant,

$$(\xi-x)\,x_1+(\eta-y)\,y_1+(\zeta-z)\,z_1=0 \quad\ldots\ldots\ldots(4),$$

$$(\xi-x)\,x_2+(\eta-y)\,y_2+(\zeta-z)\,z_2=a \quad\ldots\ldots\ldots(5),$$

$$(\xi-x)\,x_3+(\eta-y)\,y_3+(\zeta-z)\,z_3=3h \quad\ldots\ldots\ldots(6).$$

The equations (4), (5), and (6) give the centre of spherica curvature. Solving, we get three equations of the type

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} (\xi-x)=a\,(y_3z_1-y_1z_3)+3h\,(y_1z_2-y_2z_1).$$

Squaring and adding, we get, by use of the identities (1),

$$\Delta R^2=a^2B+6ahF+9h^2C \quad\ldots\ldots\ldots\ldots(7),$$

where $$\Delta\equiv\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}\equiv\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}^2.$$

This is the expression for spherical curvature in terms of the fundamental quantities.

3. Let (lmn) be the actual direction-cosines of the tangent, $(\lambda\mu\nu)$ of the binomial.

Then we have

$$l=x_1a^{-\frac{1}{2}},$$

therefore $$\frac{dl}{dt}=x_2a^{-\frac{1}{2}}-a^{-\frac{3}{2}}x_1h \quad \text{by (2)}$$

$$=a^{-\frac{3}{2}}\,(ax_2-hx_1),$$

and two similar equations. Squaring and adding, we have, with the usual notation,

$$\left(\frac{d\psi}{dt}\right)^2=\frac{a^2b+h^2a-2ah^2}{a^3}=\frac{C}{a^2},$$

also $\left(\dfrac{dt}{ds}\right)^2 = \dfrac{1}{a}$, so that we get

$$\frac{1}{\rho^2} = \frac{C}{a^3} \quad\text{..........................(8).}$$

Again, the binormal is perpendicular to two consecutive tangents, and therefore we have $\Sigma\lambda x_1 = \Sigma\lambda x_2 = 0$.

Thus $$\lambda = (y_1 z_2 - y_2 z_1)\, C^{-\frac{1}{2}}.$$

Differentiating, we have by means of (2)

$$\frac{d\lambda}{dt} = \frac{F(y_1 z_2 - y_2 z_1) - C(y_3 z_1 - y_1 z_3)}{C^{\frac{3}{2}}}$$

on reduction.

Squaring and adding to the similar equations, we get

$$\left(\frac{d\tau}{dt}\right)^2 = \frac{1}{C^3}\left[C^2 B + CF^2 - 2CF^2\right] = \frac{\Delta a}{C^2},$$

so that $$\frac{1}{\sigma^2} = \frac{\Delta}{C^2} \quad\text{..........................(9).}$$

From (8) we have

$$2\rho\frac{d\rho}{dt} = \frac{6a^2 h C + 2a^3 F}{C^2} = \frac{2a^2}{C^2}(aF + 3hC),$$

therefore $$\left(\frac{d\rho}{dt}\right)^2 = \frac{a}{C^3}(aF + 3hC)^2.$$

Combining this with $\left(\dfrac{dt}{d\tau}\right)^2 = \dfrac{C^2}{\Delta a}$ we get

$$\left(\frac{d\rho}{d\tau}\right)^2 = \frac{(aF + 3hC)^2}{\Delta C} \quad\text{................(10).}$$

From (10) we can easily prove the formula

$$R^2 = \rho^2 + \left(\frac{d\rho}{d\tau}\right)^2.$$

For we have

$$\rho^2 + \left(\frac{d\rho}{d\tau}\right)^2 = \frac{\Delta a^3 + (aF + 3hC)^2}{\Delta C}$$

$$= \frac{a^2(BC - F^2) + a^2F^2 + 6ahCF + 9h^2C^2}{\Delta C}$$

$$= R^2 \quad \text{from (7).}$$

4. The preceding expressions are, of course, mathematically identical with those given in most standard treatises. But the fundamental quantities usually taken* are six of the forms

$$X = dyd^2z - dzd^2y,$$

$$X' = dyd^3z - dzd^3y.$$

It is, however, I think, worthy of notice that it is possible to replace these by an invariantive set of quantities derived from the determinant which is the square of

$$\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix}.$$

Again, since the differentiation of these functions leads to similar fundamental functions of the fourth order, it is evident that variations of torsion and spherical curvature can be expressed in terms of them, though the actual expressions are so complicated as to render the finding of them of little interest. And, in general, any quantity connected intrinsically with the curve, and involving considerations of n consecutive points, can be expressed in terms of fundamental quantities of order up to n.

* Compare H. v. Mangoldt's paper in the *Encyclopaedie der Math Wissenschaften* (Bd. III., 3, Heft 1, p. 82, et seq.), and Salmon, *Geometry of Three Dimensions*, 3rd edition, p. 328.

ON LINKS AND KNOTS IN EUCLIDEAN SPACE OF n DIMENSIONS.

By *Duncan M. Y. Sommerville, M.A., D.Sc.*

In 1876 Klein showed that knots cannot exist in space of four dimensions,* and Hoppe† and Durège‡ have given examples of the resolution of an ordinary "insoluble" knot by motion in 4-space. As I have not come across any investigation of the existence of knots in space of higher dimensions than four, the following treatment of the problem from elementary considerations may be of interest.

An ordinary knot may be considered as a system of links in a continuous closed curve. If we consider two branches as linked together, the removal of this link will in general simplify the knot, and by successively unlinking all the branches the knot will finally be resolved. If then we can show that two linked rings can be unlinked by a motion in space of higher dimensions, it follows that any knot can be resolved. This is the known result that ordinary knots, *i.e.* knotted curves, can all be resolved in space of *four* dimensions. Again in the plane, or space of two dimensions, two closed curves cannot link without intersecting, and so, for an apparently quite different reason, knots cannot exist in space of *two* dimensions. The explanation of the apparent difference in the reasons for the two similar results is to be found in the conception of a knot. The ordinary conception of a knot is a closed curve whose plane projections are all closed curves with (three or more) double points, however the curve may be transformed under the conditions that it remains always closed and never cuts itself, or, in Tait's notation, it is a curve whose plane projections are always closed curves with double points, and having an equation of the form

$$\rho = \phi(s),$$

ϕ being a periodic function, whose variations are subject to the condition that "*no two values of ρ shall ever be equal* even

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* *Math. Ann.*, Vol. IX., p. 478.

† *Arch. der Math.*, Vol. LXIV., p. 224 (1879); Vol. LXV., pp. 423–426 (1880).

‡ *Wien. Ber.*, Vol. LXXXII., pp. 135–146 (1880). Cf. also Simony, "Gemeinfassliche, leicht controlirbare Lösung der Aufgabe: 'In ein ringförmig geschlossenes Band einen Knoten zu machen' und verwendter merkwürdiger Probleme." Wien, (Gerold), 3rd ed., 1881; and Schlegel, *Schlömilch Zs.*, Vol. XXVIII, pp. 105–115 (1883).

at a *stage* of the deformation."* With this definition it can be shown that knots only exist in space of three dimensions. We shall see, however, that in space of five dimensions it is possible to obtain a closed *surface* satisfying analogous conditions, and therefore forming the true analogue of a knot. We shall therefore define a knot in Euclidean space of n dimensions as a closed surface of m dimensions whose projections in space of $m+1$ dimensions are all closed nodal surfaces for all possible deformations of the surface subject to the conditions that it is always closed and never cuts itself. The relation between m and n in order that such a surface should exist will be the subject of our investigation.

In space of n dimensions R_n consider a hypersphere S_{r-1} in R_r. If r points of this R_r are fixed it may rotate about the R_{r-1} determined by these points. Each point will then describe an S_{n-r}, and the whole S_{r-1} will describe a surface of revolution of the fourth order, which is called a *tore* of the $(n-r)^{\text{th}}$ *power*. Denote this by $T^4_{n,n-r}$, where we may for simplicity omit the 4. An ordinary anchor ring is $T^4_{3,1}$.

Take $r-1$ rectangular axes, $x_1, \ldots, x_{r-1}$, in the R_{r-1}, which we shall call the *axle* of the tore, not to confound it with the coordinate *axes*, and one other, x_r, passing through the centre of the generating S_{r-1}. The coordinates of the centre of the S_{r-1} are then $0, \ldots, 0, a$, and its equation referred to these axes is

$$(x_r - a)^2 + \sum_1^{r-1} x_\rho^2 = b^2;$$

whence $$x_r = a \pm \sqrt{(b^2 - \sum_1^{r-1} x_\rho^2)}.$$

Now take other axes in R_n, with the same origin, so as to make with the r axes already chosen a system of n rectangular axes. The equation of the $T_{n,n-r}$ is then

$$x_r^2 + \sum_r^{n-1} x_\rho^2 = \{a \pm \sqrt{(b^2 - \sum_1^{r-1} x_\rho^2)}\}^2,$$

or $$(\sum_1^n x_\rho^2 - a^2 - b^2)^2 = 4a^2 (b^2 - \sum_1^{r-1} x_\rho^2),$$

* *Proc. R.S. Edin.*, Vol. IX., p. 403.

which may also be written

$$(\sum_{1}^{n} x_p^2 + a^2 - b^2)^2 = 4a^2 \sum_{r}^{n} x_p^2.$$

If we cut this by any hyperplane such as $x_1 = 0$ perpendicular to the axle we get a $T_{n-1, n-r}$.

If we cut it by any hyperplane passing through the axle, such as $x_n = 0$, we get a $T_{n-1, n-r-1}$.

Generally, if we cut it by the hyperplanes

$$x_1 = 0, \quad x_2 = 0, \ \ldots, \ x_p = 0 \qquad (p \not> r-1),$$

$$x_n = 0, \ x_{n-1} = 0, \ \ldots, \ x_{n-q+1} = 0 \quad (q \not> n-r+1),$$

we get a tore in which n is to be replaced by $n-p-q$ and r by $r-p$, *i.e.* a $T_{n-p-q, n-q-r}$.

The highest power of a tore in R_n is $n-1$; such a tore consists of a pair of concentric hyperspheres, $\sum_{1}^{n} x_p^2 = (a \pm b)^2$. A tore $T_{n,n-r}$ will be cut in a tore of highest power by any hyperplane perpendicular to the axle and of dimensions $\not> n-r+1$.

The lowest power of a tore in R_n is 0; such a tore consists of a pair of hyperspheres, $\sum_{1}^{n-1} x_p^2 + (x_n \pm a)^2 = b^2$. A tore $T_{n,n-r}$ will be cut in a tore of lowest power if $q = n-r$, *i.e.* by any hyperplane which contains completely the axle of the tore.

Consider now two tores $T_{n,n-r}$, $T'_{n,n-s}$ of equal radii, and such that $a > 2b$, and let them be placed with their centres at a distance a apart and the two sets of rectangular axes through their centres parallel. Let the axle of the first be determined by the axes of

$$x_1, x_2, \ldots, x_p, x_{p+1}, \ldots, x_{r-1},$$

and let the other one be for the moment placed with its centre also at the origin and its axle determined by the axes of

$$x_1, x_2, \ldots, x_p, x_r, \ldots, x_{r+s-p-2}.$$

Then, moving its centre along the axis of x_n through a distance a, the equations of the axle become

$$x_n = a, \ x_{n-1} = 0, \ \ldots, \ x_{r+s-p-1} = 0.$$

The minimum dimensions of a hyperplane which can be drawn through both the axles is then

$$q = r - 1 + s - 1 + 1 - p = r + s - p - 1.$$

In their present positions the tores may cut one another, or they may be linked, either separably or inseparably. In particular if the tores are each of highest power in the same space R_n they will necessarily intersect, except in the particular case $n = 1$, when each tore reduces to a pair of points, and forms also a tore of *lowest* power.

Let each tore be cut by the hyperplane of $n - q + 1$ dimensions

$$x_1 = 0, \ldots, x_p = 0, x_{p+1} = 0, \ldots, x_{r-1} = 0, x_r = 0, \ldots, x_{r+s-p-2} = 0.$$

The first is cut in a

$$T_{n-(r-1)-(s-1-p),\, n-(s-1-p)-r}, \ i.e. \text{ a } T_{n-r-s+p+2,\, n-s-r+p+1};$$

and the second in a

$$T_{n-(s-1)-(r-1-p),\, n-(r-1-p)-s}, \ i.e. \text{ a } T_{n-r-s+p+2,\, n-r-s+p+1},$$

or each is cut in a $T_{n-q+1,\, n-q}$, *i.e.* a tore of highest power in R_{n-q+1}. These will necessarily intersect if $n - q > 0$ or if $q < n$.

Hence two tores placed in this way must pierce each other if a hyperplane of $n - 1$ dimensions or less can be drawn to contain both their axles.

Let the tores be placed concentric, so that as few as possible of the axes in their axles coincide. There must be $(r-1) + (s-1) - n$ coincident, hence, if $r + s - 2 \geqq n$, the least value of p is $r + s - 2 - n$. The two axles therefore determine in this case $r - 1 + s - 1 - (r + s - 2 - n) = n$, *i.e. all* the axes. Hence, if $r + s > n + 1$ the tores do not necessarily intersect.

If $r + s \leqq n$, p may $= 0$, and the two axles determine $r - 1 + s - 1 = r + s - 2$ of the axes. The minimum dimensions of a hyperplane through both axles is then $r + s - 2 + 1 = r + s - 1$, which is $\leqq n - 1$, and therefore the tores must intersect.

If $r + s - 1 = n$, p may $= 0$ and the two axles determine $r - 1 + s - 1 = r + s - 2 = n - 1$ of the axes. The minimum dimensions of a hyperplane through both axles is then $n - 1 + 1 = n$, and the tores do not necessarily intersect.

Hence, if $r + s \leqq n$ the tores must intersect; if $r + s > n$ they need not intersect.

If $r = 1$ the first tore becomes a pair of concentric hyperspheres and must be cut by the other tore if $s < n$. If $s = n$

the other tore becomes a pair of hyperspheres, one within and one outside the pair of concentric hyperspheres, and they do not intersect. In this case the tores are, of course, inseparably linked. Let us investigate the condition that the tores may be inseparably linked. It is a necessary condition that one or more of the sections by coordinate planes should be tores inseparably linked. Only those sections which contain the axis of x_n will be both tores. Of these sections there are $n-1$. If one of these only is inseparably linked, say the section $x_1=0$, motion along all the axes but that of x_1 is restricted, while motion along this axis is quite free and the tores are separable. If two of the sections, say $x_1=0$, $x_2=0$, are inseparable, motion along all the axes is restricted, all the $n-1$ sections are inseparable and the tores are inseparable. It is a necessary and sufficient condition therefore that any two of the sections containing the axis of x_n, say $x_1=0$ and $x_2=0$, should be inseparable. For this, again, it is necessary and sufficient that the sections

$$(x_1=0,\ x_2=0)\ (x_1=0,\ x_3=0)\ (x_2=0,\ x_3=0)$$

should be inseparable, hence also for the sections

$$(x_1,\ x_2,\ x_3=0)\ (x_1,\ x_2,\ x_4=0)\ (x_1,\ x_3,\ x_4=0)\ (x_2,\ x_3,\ x_4=0),$$

and so on; so that finally we find that it is necessary and sufficient that the section by every two-dimensional plane passing through the axis of x_n should be inseparable; *i.e.* every such section must consist of a pair of distinct circles, and a pair of concentric circles surrounding one of these.

The section of the first tore by the two-dimensional plane $(x_1, x_2, \ldots, x_{n-2}=0)$ is a $T_{n-(r-1)-(n-2-r+1),\, n-(n-2-r+1)-r}$, or a $T_{2,1}$. The section by $(x_2, \ldots, x_{n-1}=0)$ is a $T_{n-(r-2)-(n-2-r+2),\, n-(n-2-r+2)-r}$, or a $T_{2,0}$.

The section of the second by $(x_1, x_2, \ldots, x_{n-2}=0)$ must be a $T_{2,0}$. The axis of x_{n-1} must therefore belong to the axle; similarly x_1 does not belong to the axle, and so on.

Hence, if the axle of the first contains $x_1, x_2, \ldots, x_{r-1}$, the other must contain $x_r, x_{r+1}, x_{r+2}, \ldots, x_{n-1}$, *i.e.* $s-1=n-r$, or $r+s=n+1$.

Hence we have the result that if

$$r+s<n+1,$$

the tores must intersect; if

$$r+s=n+1,$$

they may be inseparably linked without intersecting; and if

$$r+s>n+1,$$

they either intersect or are separable.

If n is odd we may have two tores of the same power inseparably linked; then $r=s=\frac{1}{2}(n+1)$; if n is even this is impossible.

If $b=0$ the tores reduce to hyperspheres. From

$$\left(\sum_1^n x_\rho^2-a^2-b^2\right)^2=4a^2\left(b^2-\sum_1^{r-1} x_\rho^2\right),$$

we get
$$\left(\sum_1^n x_\rho^2-a^2\right)^2+4a^2\sum_1^{r-1} x_\rho^2=0,$$

therefore
$$x_1=0,\ \ldots,\ x_{r-1}=0,$$

$$\sum_1^n x_\rho^2=a^2.$$

Hence we conclude that in R_n two closed surfaces, each of $\frac{1}{2}(n+1)$ dimensions, may be inseparably linked without intersecting.

By supposing the two surfaces to join to form a continuous closed surface linking itself, we get the following results:—

In R_n a continuous closed surface of m dimensions may be properly knotted if n is odd and $m=\frac{1}{2}(n+1)$. If $m<\frac{1}{2}(n+1)$ the knot is resoluble, if $m>\frac{1}{2}(n+1)$ the surface necessarily cuts itself.

If n is even, a knot is impossible, m being either $<\frac{1}{2}(n+1)$ or $>\frac{1}{2}(n+1)$, *i.e.* either the knot is resoluble or the surface cuts itself.

Of course, a proper knot in R_{2n-1} can always be resolved in R_{2n}, and its projections in space of lower dimensions are self-intersecting.

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HIGH QUARTAN FACTORISATIONS AND PRIMES.

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[The author is indebted to Mr. H. J. Woodall, A.R.C.Sc., for help in reading the Proof-sheets and for many useful suggestions.]

1. *Quartans, Octavans.* THE numbers discussed in this Paper are of forms

$$N = x^4 + y^4, \text{ and } N = x^8 + y^8 \ldots\ldots\ldots\ldots\ldots (1).$$

For shortness' sake these algebraic forms will be styled *Quartans*, and *Octavans*, respectively.

1a. *Working condition.* For sake of brevity the *working condition* of "x prime to y" is generally assumed. With this condition N or $\frac{1}{2}N$ is always odd; in fact

x, y one odd, one even, give N *odd*........................(1a),

x, y both odd give N even, and $\frac{1}{2}N$ odd...... ....(1b).

In the latter case $\frac{1}{2}N$ will be styled a *Half-Quartan** or *Half-Octavan.**

2. *Notation.* All symbols denote *integers*;

ω, Ω denote *odd* numbers; ϵ, e denotes *even* numbers; i *any* integer; p denotes a *prime*; N is defined in (1).

3. *Linear Forms.* The following simple linear forms of N and $\frac{1}{2}N$ to various moduli are to be noted as occurring under the "conditions" named:

Condition.	*Quartans.*	*Half-Quartans.*	*Octavans.*	*Half-Octavans.*
None	$16n+1$	$8n+1$	$32n+1$	$16n+1$.......(2a),
x or $y=3i$	$16.3n+1$	$8.3n+1$	$32.3n+1$	$16.3n+1$........(2b),
x or $y=5i$	$16.5n+1$	$8.5n+1$	$32.5n+1$	$16.5n+1$........(2c),
x or $y=5i$	$100n+\epsilon,1$†	$100n+\omega,3$†	$100n+\epsilon,1$†	$100n+\omega,3$† ...(2d),
x or $y \neq 5i$	$100n+\omega,7$†	$100n+\epsilon,1$†	$100n+\omega,7$†	$100n+\epsilon,1$† ....(2e),
$x=\omega_1\omega_2$,	$m(2\omega_1)^4+\omega_1^4+y^4$,	$m.2^3\omega_1^4+\frac{\omega_1^4+y^4}{2}$,	$m.2^5\omega_1^8+\omega_1^8+y^8$,	$m.2^4\omega_1^8+\frac{\omega_1^8+y^8}{2}$..(2$f$).

* It is proposed to reserve the terms *Semi-Quartan*, and *Semi-Octavan* for the forms $N=x^2+y^4$, $N=x^4+y^8$, respectively.

† Here ϵ, 1; ω, 3; ω, 7 are to be read *arithmetically* as (the tens and units) digits of N and $\frac{1}{2}N$.

4. *Quadratic Forms* (of Quartans). Every Quartan (N) and Half-Quartan ($\frac{1}{2}N$) may be expressed (algebraically) in the following four quadratic forms

$$N = a^2 + b^2 = (x^2)^2 + (y^2)^2 \dots\dots (3a),$$

$$= c^2 + 2d^2 = (x^2 \sim y^2)^2 + 2(xy)^2 \dots\dots (3b),$$

$$= e^2 - 2f^2 = (x^2 + y^2)^2 - 2(xy)^2 \dots\dots (3c),$$

$$= 2f'^2 - e'^2 = 2(x^2 \mp xy + y^2)^2 - (x^2 \mp 2xy + y^2)^2 \dots (3d),$$

$$\tfrac{1}{2}N = a^2 + b^2 = \left(\frac{x^2+y^2}{2}\right)^2 + \left(\frac{x^2 \sim y^2}{2}\right)^2 \dots\dots (4a),$$

$$= c^2 + 2d^2 = \quad (xy)^2 \quad + 2\left(\frac{x^2 \sim y^2}{2}\right)^2 \dots\dots (4b),$$

$$= e^2 - 2f^2 = (x^2 \mp xy + y^2)^2 - 2\left(\frac{x^2 \mp 2xy + y^2}{2}\right)^2 \dots (4c),$$

$$= 2f'^2 - e'^2 = 2\left(\frac{x^2+y^2}{2}\right)^2 - (xy)^2 \dots\dots (4d).$$

These will be styled—for shortness—the (a, b), (c, d), (e, f), (e′, f′) partitions. Note that—disregarding signs—

$$\text{In } N;\ 2a \text{ or } 2b = c + e,\ d = f = f' \sim e' \dots\dots (5a),$$

$$\text{In } \tfrac{1}{2}N;\quad a \text{ or } b = d,\ b \text{ or } a = f',\ c = e' \dots\dots (5b).$$

When N or $\frac{1}{2}N$ is odd, the "terms" of each partition are one odd, and one even; it is usual to take

$$a, c, e, e' \textit{ odd};\ b, d, f, f' \textit{ even} \dots\dots (6).$$

The (e, f), (e′, f′) partitions, having the same determinant ($D = +2$), are not to be considered *different forms*, being merely different expressions of the *same form*; they are in fact immediately interconvertible by the relations

$$e' = e \mp 2f,\ f' = e \mp f;\ e = 2f' \mp e',\ f = f' \mp e' \dots\dots (7).$$

The (a, b), (c, d), (e, f) forms, having different determinants ($D = -1, -2, +2$) are to be considered *different forms*: they are however not independent, but are so connected that any one of them may be derived* from the other two.

* See Art. 24 to 36 of the author's Paper on *Connexion of Quadratic Forms* in *Proc. Lond. Math. Soc.*, Vol. XXVIII, 1896.

When N or $\frac{1}{2}N$ is prime, each of the (a, b), (c, d) forms is *unique*: the (e, f), (e′, f′) forms may also be considered *unique*; for, although N and $\frac{1}{2}N$ may be expressed in an infinite number of ways in these forms, yet all these expressions are mere *automorphs* of the (e, f), (f′, e′) forms given above, which are in fact their *Base-forms*, *i.e.* the forms with minimum values (when the upper signs are used) of e, f, e′, f′; the Base-forms are marked by the property—

$$e > 2f, \quad f' > e' \quad \text{.........................} (7a).$$

When N or $\frac{1}{2}N$ is composite, and contains different primes as factors (so as not to be merely a power of a single prime), it is always expressible—but not as a rule algebraically (unless the partitions of the component factors be known)—in more than one way in each of the above 2^ic forms. Each resolution into a pair of (unequal) factors gives rise to a different base-form in the (e, f) and (f′, e′) partitions; but it is beyond the scope of this Paper to enter into these merely arithmetical ways of 2^ic partition.

4*a*. *Expression of given* 2^ic *forms as Quartans, &c.* When an *odd* number N or $\frac{1}{2}N$ is given in some *one* of the 2^ic forms (a, b), (c, d), (e, f), (e′ f′), it will be expressible as a Quartan or Half-Quartan with elements (x, y) as below, provided the quantities under the radicals are *perfect squares*; (this involves *two conditions* in each case):—

$$N = a^2 + b^2\,; \quad x = \sqrt{a}, \quad y = \sqrt{b} \quad \text{..................} (3a'),$$

$$N = c^2 + 2d^2\,; \quad x \text{ or } y = [\tfrac{1}{2}(\sqrt{c^2 + 4d^2} \pm c)]^{\frac{1}{2}} \quad \text{..................} (3b'),$$

$$N = e^2 - 2f^2\,; \quad x \text{ or } y = [\tfrac{1}{2}(\sqrt{e^2 - 4f^2} \pm e)]^{\frac{1}{2}} \quad \text{..................} (3c'),$$

$$N = 2f'^2 - e'^2\,; \quad x \text{ or } y = \tfrac{1}{2}[\pm\sqrt{e'} \pm \sqrt{4f' - 3e'}] \quad \text{..................} (3d'),$$

$$\tfrac{1}{2}N = a^2 + b^2\,; \quad x = (a \pm b)^{\frac{1}{2}}, \; y = (a \mp b)^{\frac{1}{2}} \quad \text{..................} (4a'),$$

$$\tfrac{1}{2}N = c^2 + 2d^2\,; \quad x \text{ or } y = [\sqrt{c^2 + d^2} \pm d]^{\frac{1}{2}} \quad \text{..................} (4b'),$$

$$\tfrac{1}{2}N = e^2 - 2f^2\,; \quad x \text{ or } y = \tfrac{1}{2}[\pm\sqrt{2f} \pm \sqrt{4e - 6f}) \quad \text{..................} (4c'),$$

$$\tfrac{1}{2}N = 2f'^2 - e'^2\,; \quad x \text{ or } y = [\sqrt{f'^2 - e'^2} \pm f']^{\frac{1}{2}} \quad \text{..................} (4d').$$

Note that the forms (e, f), (e′, f′) to be used in Results (3*c*′), (3*d*′), (4*c*′), (4*d*′) must be their *Base-forms*, (see the requisite condition (7*a*)).

4*b*. *Quadratic Forms* (of Octavans). Octavans and Half-Octavans may be expressed (algebraically) in the same 2^ic

forms as the Quartans and Half-Quartans of which they are merely specialised forms. All the results of Art. 4 are applicable to them also, except that x^2, y^2 must be substituted for the x, y of those formulæ.

5. *Divisors.* When x, y are both odd, then N is *even*, and has the divisor 2 (but not 4), so that $\frac{1}{2}N$ is always *odd* (cf. Art. 1*a*). All other divisors are either primes of the forms

$$p = 8\varpi + 1 \text{ for Quartans; } p = 16\varpi + 1 \text{ for Octavans} \qquad \ldots\ldots\ldots\ldots (8);$$

or powers of such primes, or products of such primes and their powers.

5*a*. 2^{ic} *forms of divisors.* Every odd divisor (Q) of N or $\frac{1}{2}N$ is expressible—but not, as a rule, algebraically—in *each* of the above 2^{ic} forms of Art. 4,

$$Q = \mathrm{a}^2 + \mathrm{b}^2 = \mathrm{c}^2 + 2\mathrm{d}^2 = \mathrm{e}^2 - 2\mathrm{f}^2 = 2\mathrm{f}'^2 - \mathrm{e}'^2 \ldots\ldots (8a),$$

and these partitions are all *unique* (in the sense above explained, Art. 4) in the case of prime divisors.

5*b*. 2^{ic} *forms of large divisors.* In the case of composite Quartans and Octavans, and their halves, say N or $\frac{1}{2}N = q \cdot Q$, the (a, b), (c, d), (e, f), (f′, e′) forms of either factor (say Q) can always be formed by the process of *conformal* division* when the corresponding 2^{ic} forms of the other factor (q) are known, provided the latter factor be either a *prime*, or a power of a prime (and can sometimes also be found by that process when q is composite).

When N is large, and has one prime, or power of a prime, divisor ($q = p$ or p^κ) so small that its 2^{ic} parts (a_1, b_1, &c.) are either known, or can be easily found, this process affords an *easy way* of finding the 2^{ic} parts of the other factor (Q) *even when very large*: this is important, as the direct determination of the 2^{ic} partitions of a very large number (as Q) is usually pretty laborious.

Ex. Take $N = 2^{64} + 1 = q\,Q$; where $q = 274177$.

The 2^{ic} partitions of the small factor q are easily found (by trial). Hence by (3*a* to *d*),

$$Q = \frac{N}{q} = \frac{1^2 + (2^{32})^2}{89^2 + 516^2} = \frac{(2^{32}-1)^2 + 2.(2^{16})^2}{523^2 + 2.28^2} = \frac{(2^{32}+1)^2 - 2.(2^{16})^2}{575^2 - 2.148^2}.$$

* *Conformal Division* is division with preservation of 2^{ic} form. For the details of this process, and for the conditions for carrying it out successfully when q is composite, see the author's Paper on *Connexion of Quadratic Forms* above quoted, Art. 15 to 23.

The process of conformal division* now gives *directly* the required partitions of the large factor $Q = 67280421310721$, viz.

$$Q = (8083111^2 + 1394180^2) = (8192757^2 + 2.282094^2) = 9007423^2 - 2.2631848^2,$$

and the last one gives also $Q = (2.6375575^2 - 3743727^2)$. These results would be difficult to obtain directly.

6. *Factorisation-Tables.* The author has had 8 Tables of the factorisation of these numbers compiled, as below (Art. 6*a*, *b*):—

6*a*. *General Tables.* The factorisation of *all* such numbers (x, y, both varying) has been worked out *completely*†‡ up to the limits named in the Abstract below, which shows also the total number (n) of each class factorised.

	N or $\frac{1}{2}N =$	(x^4+y^4)	$\frac{1}{2}(x^4+y^4)$	x^8+y^8	$\frac{1}{2}(x^8+y^8)$
Limits of	$x =$	$\omega \not> 53$	$\omega \not> 65$	$\omega \not> 11$	$\omega \not> 11$
	$y =$	$\varepsilon \not> 54$	$\omega \not> 55$	$\varepsilon \not> 10$	$\omega \not> 9$
	N or $\frac{1}{2}N \not>$	9 million	9 million	215 million	107 million
	Number (n)	534	367	26	11

The Quartan factorisation was done by the large Factor-Tables, and therefore extends only to their limit (9 million). The Octavans being few in number, the factorisation was carried much further by special means.

6*b*. *Simple* 4-*tan and* 8-*van Tables* ($x = 1$). The factorisation of all *simple* Quartans and Octavans, $N = (1 + y^4)$, and $(1 + y^8)$, has been carried out§‖ as completely as possible with the means available (to be described in Art. 8 to 15) up to the high limits named in the Abstract below, which shows also the degree of success attained, *i.e.* the number of each class completely or partially factorised.

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* See footnote * above.

† The 4-tan Tables by Miss E. Cooper, and checked by Miss A. Woodward, under the author's superintendence. The 8-van Tables by the author himself.

‡ These Tables are not published: the 4-tan Tables could be readily prepared by any computer (from the large Factor-Tables).

§ These 4-tan Tables were computed by the late Mr. C. E. Bickmore and the present author jointly, as far as $y = 100$. The 4-tan Tables beyond $y = 100$, and the 8-van Table, were computed partly by the author himself, partly by two Assistants (Miss A. Cole and Miss E. Cooper), and checked throughout by one of them and in part also by another Assistant (Miss A. Woodward), under the author's superintendence.

‖ These Tables are thought too extensive for publication herewith: it is hoped to publish them in a separate Work.

	Complete Factoris'n.		Factorisation.				Limit.	
N or $\frac{1}{2}N$; y	Range of y;	Number	Range of y;	Compl.	Partl.	Fail	Total.	N or $\frac{1}{2}N$
$(1+y^4)$; ε	2 to 226;	113 (All)	228 to 1000;	219,	63;	105	500;	10^{12}
$\frac{1}{2}(1+y^4)$; ω	1 to 265;	133 (All)	267 to 999;	230,	35;	1 2	500;	$\frac{1}{2}10^{12}$
$(1+y^8)$; ε	2 to 18;	9 (All)	20 to 160;	16,	42;	13	80;	4.10^{17}
$\frac{1}{2}(1+y^8)$; ω	1 to 19;	10 (All)	21 to 159;	16,	38;	16	80;	2.10^{17}

7. *Quartan, &c., Octavan, &c., Primes, and Factors;* (Tab. I. to VII.). These Tables give *complete lists* of all Quartan, Half-Quartan, Octavan, and Half-Octavan primes, and also of all High Prime Factors ($p>9$ million) of simple Quartans, Half-Quartans, Octavans, and Half-Octavans, up to the high limits named in the Abstract below, and also a few beyond those limits: the Abstract shows also the *number* of primes of each class within the limits stated; those beyond the limits (of x, y or p) stated are numbered as "Extra."

The Quartan results are classified in six Tables (Tab. I.–VI.) as in the Abstract below: the Octavan results are similarly classified, but (being few in number) are all printed in one Table (Tab. VII.). Inasmuch as Octavans are merely specialised Quartans, some few of the Octavan results appear in Tab. I.–VI. (when comprised within the limits of these Tables) as well as in Tab. VII.

	Tab.	p	Limits of p	x	y	Numb.	Extra
Quartans & Half-Quartans	I.	(x^4+y^4)	$\not>9$ million	$\omega \not> 53$	$\varepsilon \not> 54$	232	.
	II.	$\frac{1}{2}(x^4+y^4)$	$\not>9$ million	$\omega \not> 65$	$\omega \not> 55$	166	.
	III.	$(1+y^4)$	$>9.10^6$, $\not> 27.10^8$	1	$\varepsilon>54$, $\not>226$	20	2
	IV.	$\frac{1}{2}(1+y^4)$	$>9.10^6$, $\not> 25.10^9$	1	$\omega>65$, $\not>265$	18	.
Factors	V.	$\frac{1}{\mu}\cdot(1+y^4)$	$>9.10^6$, $\not> 10^9$	1	$\varepsilon>110$, $\not>1000$	108	5
	VI.	$\frac{1}{\mu}\cdot\frac{1}{2}(1+y^4)$	$>9.10^6$, $\not> 10^9$	1	$\omega>131$, $\not>999$	102	4
Octavans & Half-Octavans	VII.	(x^8+y^8)	$\not>9$ million	$\omega \not> 7$	$\varepsilon \not> 6$	3	.
	,,	$\frac{1}{2}(x^8+y^8)$	$\not>9$ million	$\omega \not> 7$	$\omega \not> 7$	2	.
	,,	(x^8+y^8)	$>9.10^6$, $\not> 13.10^{11}$	1	$\varepsilon>6$, $\not>36$	.	1
	,,	$\frac{1}{2}(x^8+y^8)$	$>9.10^6$, $\not> 18.10^9$	1	$\omega>7$, $\not>19$	2	1
Factors	,,	$\frac{1}{\mu}\cdot(1+y^8)$	$>9.10^6$, $\not> 10^9$	1	$\varepsilon>10$, $\not>160$	9	.
	,,	$\frac{1}{\mu}\cdot\frac{1}{2}(1+y^8)$	$>9.10^6$, $\not> 10^9$	1	$\omega>11$, $\not>159$	8	.

Divisor μ. Note that in Tab. V., VI., VII., the entry μ (in the column headed μ) indicates that the divisor (μ) needed is >100.

Nearly all the primes here reported were detected in the course of the factorisation described in Art. 6*a*, *b*: the magni-

tude of the High Primes is therefore generally limited by the extent of those Tables, and by the means of factorisation available (Art. 8), *i.e.*

Factorisation-Limit.	*High Prime-Limit.*
In 4-tans, $y \ngtr 1000$; In 8-vans, $y \ngtr 160$	Usual, $p \ngtr 105.10^7$; Special, $p \ngtr 26.10^8$.

The few marked "Extra" in the Abstract above were either specially worked out by, or (in a few cases only) obtained from other Works.

7*a*. *High Primes.* An Abstract of the magnitudes of the High Primes here reported is given below. Most of them are believed to be *new* (*i.e.* not previously published): those previously published—so far as known to the author—are marked in the Tables (Tab. III. to VII.) by a capital letter (B, D, &c.) which serves to indicate the name of the discoverer and Work in which published according to the scheme in the footnote.*

p	7-fig.	8-fig.	9-fig.	10-fig.	Total	p	7-fig.	8-fig.	9-fig.	10-fig.	Total
x^4+y^4	1,	5,	11,	5;	22	x^8+y^8	.	.,	1,	.;	1
$\frac{1}{2}(x^4+y^4)$	.,	7,	9,	2;	18	$\frac{1}{2}(x^8+y^8)$	.	1,	2,	.;	3
$\frac{1}{\mu}(x^4+y^4)$	2,	69,	42,	.;	113	$\frac{1}{\mu}(x^8+y^8)$	.	3,	6,	.;	9
$\frac{1}{\mu}.\frac{1}{2}(x^4+y^4)$	6,	65,	34,	1;	106	$\frac{1}{\mu}.\frac{1}{2}(x^8+y^8)$	.	4,	4,	.;	8
Total	9,	146,	96,	8;	259	Total	.	8,	13,	.;	21

7*b*. *Few prime binomials of high order.* The number of primes which are simple binomials, or half-binomials of even order, *i.e* of form

$$p=(1+y^e), \text{ or } p=\tfrac{1}{2}(1+y^e), \text{ where } e=2, 4, 8, \&c.,$$

will be found to decrease rapidly as e increases: the comparison may be made in two ways, viz.

(1) within same range of y; (2) within same range of p.

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* As far as known to the author only 15 of these High Primes had been previously published, or, previously discovered by others, viz.:

2 marked D, due to E. Desmarest, see *Théorie des Nombres*, Paris, 1852, p. 286.

1 marked Lf, due to W. Looff, see *Nouv. Ann. de Mathém*, 2° Sér. t. xiv., 1885, p. 116.

1 marked Da. due to W. B. Davis, see Liouv. *Journ. de Mathém. pures et appl.*, Sér. 2°, t. xi, 1866, pp. 188—190.

4 marked L, due to F. Landry, see *Décomposition des Nombres* ($2^n \pm 1$) &c., Paris, 1869.

1 marked C, due to the author, see *Quarterly Journ. of P. & Appl. Maths.*, v. 35, 1903, p. 21.

6 marked B, due to the late C. E. Bickmore and the author jointly (not published).

1 marked J., (among the last six) confirmed by Morgan Jenkins in letter to the author.

The following Table gives the data*, *i.e.* the *number* of primes (n) of each order for same ranges of y and p; [the numbers when $e=1$ have been added for sake of comparison].

Form $p=(1+y^e)$					*Form* $p=\frac{1}{2}(1+y^e)$				
Limit of y or p; e =	1,	2,	4,	8	Limit of y or p; e =	1,	2,	4,	8
$y \not> 36$; n =	13,	10,	8,	2	$y \not> 19$; n =	5,	7,	7,	3
$y \not> 226$; n =	40,	37,	31,	?	$y \not> 265$; n =	33,	44,	33,	?
$y \not> 15000$; n =	1755,	1199,	?,	?	$y \not> 14999$; n =	951,	1288,	?,	?
$p \not> 9$ million; n =	602568†,	302,	11,	2	$p \not> 9$ million; n =	602568†,	445,	15,	2

8. *Congruence-Tables.* The factorisation of $N=(1+y^4)$ and $(1+y^8)$ up to the high limits tried (Art. 6*b*), and the detection of the High Primes reported (Art. 7*a*) were rendered possible chiefly by the preparation of extensive Tables‡§ of solutions of the two Congruences

$$y^4+1 \equiv 0 \pmod{p \text{ and } p^\kappa}, \quad \text{and } y^8+1 \equiv 0 \pmod{p \text{ and } p^\kappa} \ldots\ldots (9).$$

These Tables are now complete and continuous up to the following high limits

For y^4+1, up to $p=32441$, $p^\kappa=193^2$; For y^8+1, up to $p=9857$, $p^\kappa=97^2$,

and have been worked out also for *many* higher primes; but in this latter part the Tables are not continuous.

9. *Reduction of Fractions.* Many of the Congruence-solutions following are presented in the *form of fractions*, thus

$$y \equiv \frac{N}{D}, \pmod{M} \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where N, D, M denote *numerator*, *denominator*, and *modulus*, respectively.

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* The number (n) of primes $p=(1+y^2)$ and $\frac{1}{2}(1+y^2)$ were obtained from MS. Factorisation Tables of these forms extending to $y=15000$, compiled by the author: these are nearly ready for publication.

† This number, the total number of primes < 9 million is taken from Glaisher's *Factor-Table* for the sixth million, 1883, page 32: it is only approximate, certain errors having been found in some of the large Factor-Tables since the count was made.

‡ These Tables were prepared in part by the author himself; but for the most past by an Assistant (Miss E. Cooper) under the author's superintendence. One of the Tests described in Art. 13 was always applied; and the additions described in Result (26) were always checked by the author himself and by another Assistant (usually Miss C. Woodward).

§ These Congruence-Tables are far too extensive for publication herewith: it is hoped to publish them hereafter in a separate Work, along with the large Factorisation-Tables described in Art. 6*b*.

Hence,

$$y \equiv \frac{N \pm m.M}{D}, \text{ (mod } M) \ldots\ldots\ldots (10a).$$

To reduce this to an integer, it is only necessary to determine m so that the numerator $(N \pm mM)$ shall be divisible by D: this gives the required *integral solution* (y). This is easy when D is small, but increases in difficulty as D increases: hence, when D is *composite*, it is often convenient to resolve it into its factors, say $D = D_1 . D_2 \ldots D_r$, and to reduce each factor separately by the above process ; Thus

(1) Reduce N/D_1 to an integer, say N_1 (as above).

(2) Reduce N_1/D_2 to an integer, say N_2 (as above), and so on.

10. *Construction of Congruence-Tables.* This consists of two distinct steps for each modulus (p or p^κ).

STEP i. Finding one root (y) of the Congruences, Art 11 to 11*f*.

STEP ii. Finding the remaining roots $(y', y''$, &c., each $< p)$ from a known root (y), Art. 12.

11. STEP i. *Finding one root* (y). This may be done in a variety of ways. Several of these will be described in the following Articles (11*a* to *f*); each has its own special conveniences, as will be explained below.

1°. From a known factorisation $N = (X^4 + Y^4)$ or $(X^8 + Y^8)$......Art. 11*a*.

2°. From a known power-congruence $a^{4\xi'}$ or $a^{8\xi''} \equiv -1$ ....Art. 11*b*.

3°. From *two* known 2^{ic} partitions, either (a, b), (c, d), (e, f) .....Art. 11*c*.

4°. From a known factorisation $N = (\alpha X^2)^4 + (\beta Y^2)^4$
with a known 2^{ic} partition $(\alpha t^2 \pm \beta u^2$, or $(t^2 \pm \alpha\beta u^2)$...........Art 11*d*.

Methods 1°, 2° are general methods, applicable (with suitable change of the index n) to *any* binomial congruences $y^n \pm 1 \equiv 0$ (mod p or p^κ). Method 3° is a special method for quartans, and method 4° is a special method for octavans.

11*a*. METHOD 1°. *From a known factorisation.* If there be given

$$N = (X^4 + Y^4) \text{ or } = (X^8 + Y^8) \equiv 0 \text{ (mod } p \text{ or } p^\kappa) \ldots (11),$$

this gives at once *four* roots of either congruence in the fractional form (Art. 9)

$$y \equiv \pm X/Y, \text{ or } y \equiv \pm Y/X \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (11a),$$

for every prime (p) and prime-power (p^κ) contained in N.

This Method is very convenient, as the reduction of the fraction is easy (Art. 9), one of the terms (X, Y) being usually small: unfortunately it is limited by the powers of factorisation (which in the case of Octavans are very small). Of course, if $X=1$, then two roots ($y=\pm Y$) are given *at sight*.

Ex. Given $31^4+28^4=17.90481$; to solve $y^4+1\equiv 0$ (mod $p=90481$),

Here $\pm y \equiv \frac{31}{28} \equiv \frac{31+3.90481}{28} = \frac{271474}{28} = \frac{19391}{2} \equiv \frac{-71090}{2} = -35545$, are two roots.

11*b*. Method 2°. *From a known power-congruence.* If there be given

$$a^{4\xi'} \equiv -1, \text{ or } a^{8\xi''} \equiv -1 \text{ (mod } p \text{ or } p^\kappa) \text{.......(12)},$$

then $y=$ least residue of $\pm a^{\xi'}$ or $\pm a^{\xi''}$ (mod p or p^κ)...(12a),

are two roots of $y^4+1\equiv 0$ or $y^8+1\equiv 0$ (mod p or p^κ) respectively.

Thus any primitive root (g) of p or p^k will always suffice to give two roots (y). To save (numerical) labor, it is desirable that ξ' or ξ'' should be as small as possible; the minimum of ξ' or ξ'' is secured when $8\xi'$ or $16\xi''$ respectively $=\xi$ the Haupt-Exponent of a (*i.e.* ξ is the minimum giving $a^\xi \equiv +1$), and the base (a) should be chosen so as to have a small Haupt-Exponent (ξ). This method is convenient only when ξ' or ξ'' is small, as otherwise the numerical labor is considerable.

Ex. Given $2^{480}\equiv +1$, and $2^{240}\equiv -1$ (mod $p=23041$).
Here $2^{30}\equiv 8183$, and $2^{60}\equiv 4343$ (mod p). Hence $y=\pm 4343$, and $y=\pm 8183$ are two roots of $y^4+1\equiv 0$, $y^8+1\equiv 0$ (mod p), respectively.

11*c*. Method 3° (for $y^4+1\equiv 0$). *From two known* 2^{ic} *partitions.* If there be given *two* of

$$p \text{ or } p^\kappa = a^2+b^2 = c^2+2d^2 = e^2-2f^2 = 2f'^2 - e'^2 \text{......(13)},$$

or (by preference)

$$\left.\begin{array}{l} y_1^2+1\equiv 0 \text{ (mod } p \text{ or } p^\kappa\text{), together with } one \text{ of} \\ p \text{ or } p^\kappa = c^2+2d^2 = e^2-2f^2 = 2f'^2 - e'^2 \end{array}\right\} \text{.....(13}a\text{)},$$

then $(a\pm b)^4+4a^4\equiv 0$, and $(a\pm b)^4+4b^4\equiv 0$ (mod p or p^κ),

and $c^4\equiv 4d^4$, $e^4\equiv 4f^4$, $e'^4\equiv 4f'^4$..............(mod p or p^κ).

Hence, eliminating 4, the four roots of $(y^4+1)\equiv 0$ are given by

$$y \equiv \pm (y_1 \pm 1) y_2,\ (\text{mod } p \text{ or } p^\kappa)\ldots\ldots\ldots\ldots(14),$$

wherein y_1 may be either root of $y_1^2+1\equiv 0$ (mod p or p^κ)...(14a),

$$\text{or}\quad y_1 \equiv \text{any one of } \frac{a}{b},\ \frac{b}{a};\ \frac{de}{cf},\ \frac{cf}{de};\ \frac{de'}{cf'},\ \frac{cf'}{de'},\ (\text{mod } p \text{ or } p^\kappa)\ ..(14b),$$

$$\text{and } y_2 \equiv \text{any one of } \frac{d}{c},\ \frac{c}{2d};\ \frac{f}{e},\ \frac{e}{2f},\ \frac{f'}{e'},\ \frac{e'}{2f'},\ (\text{mod } p \text{ or } p^\kappa)\ldots(14c),$$

The above gives a great choice of formulæ for the terms y_1, y_2 entering into y. This is a very convenient method when the denominators in y_1, y_2 are small or composed of small factors, as this renders the reduction of the fractions easy (Art. 9). The reduction of y_1 can be avoided if the roots (y_1) of $y_1^2+1\equiv 0$ are known; [see (14), (14a)].

Thus, by this Method the solution of $y^4+1\equiv 0$ (mod p or p^κ) is always possible when two (independent) partitions (13) can be* found.

Ex. Given $p=99961=275^2+156^2=293^2+2.84^2$.

$$\text{Here } y \equiv \frac{a-b}{b}\cdot\frac{c}{2d} \equiv \frac{275-156}{156}\cdot\frac{293}{2.84} = \frac{4981}{32.9.13}\ (\text{mod } p)$$

$$\equiv \frac{4981+6p}{32.9.13} = \frac{604747}{32.9.13} = \frac{46519}{32.9} \equiv \frac{-53442}{32.9} = \frac{-2969}{16} \equiv \frac{-2969+p}{16}$$

$$= \frac{96992}{16} = 6062 \quad (\text{mod } p).$$

Hence $y=\pm 6062$ are two roots of y^4+1.

Or, given $p=99961=293^2+2.84^2$, and $y_1=\pm 37804$ the roots of $y_1^2+1\equiv 0$ (mod p).

Here $y \equiv (y_1-1)\cdot\dfrac{c}{2d} = 37803.\dfrac{293}{2.84} = \dfrac{12601.293}{8.7}$, which also gives $y=6062$ on reduction.

11d. Method 4° (for $y^8+1\equiv 0$). *From a known 4-tan factorisation with certain* 2^{ic} *partitions.* If there be given

$$N=(\alpha X^2)^4+(\beta Y^2)^4 \equiv 0\ (\text{mod } p \text{ or } p^\kappa,\ p=16\varpi+1;\ XY>1)..(15),$$

$$\text{with one of}\quad p=\alpha t^2 \pm \beta u^2,\ \text{or} = t'^2 \pm \alpha\beta u'^2\ldots\ldots\ldots\ldots\ldots(15a),$$

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* The author's Tables of *Quadratic Partitions*, London, 1904, give the (a, b), (c, d) partitions of all primes $p=8\varpi+1$ up to 100000: these enable $y^4+1\equiv 0$ (mod p & p^κ) to be *directly* solved to same limit.

or with one of p (or p^κ) $= \alpha t_1^2 \pm u_1^2$,

and one of p (or p^κ) $= t_2^2 \pm \beta u_2^2$...(15b),

or with 2^{ic} congruences to mod. p (or p^κ) *of same form* as the 2^{ic} partitions ;

then $\qquad \alpha^4 X^8 \equiv -\beta^4 Y^8 \pmod{p \text{ or } p^\kappa}$ ....(16a),

and $\qquad \beta^4 u^8 \equiv \alpha^4 t^8$, or $(\alpha\beta)^4 u'^8 \equiv t'^8$..............(16b),

or $\qquad \beta^4 u_2^8 \equiv t_2^8$, and $u_1^8 \equiv \alpha^4 t_1^8$.................(16c).

Eliminating α, β gives four roots (y) of $y^8 + 1 \equiv 0$ (mod p or p^κ) in the form

$$y \equiv \pm \frac{Y}{X} \cdot y', \text{ or } \equiv \pm \frac{X}{Y} \cdot \frac{1}{y'} \pmod{p \text{ or } p^\kappa} \text{......(17)},$$

where $\qquad y' \equiv$ any of $\frac{t}{u}$, $\frac{t'}{\alpha u'}$, $\frac{\beta u'}{t'}$, $\frac{t_1 t_2}{u_1 u_2}$ ....(17a),

11e. *Simple Case* ($\alpha X^2 = 1$). Writing $\alpha X^2 = 1$ in (15), reduces it to the simpler form

$$N = 1 + (\beta Y^2)^4 \equiv 0 \pmod{p \text{ or } p^\kappa} \text{........(15')}.$$

This form is important, as it enables a *known* solution of $y^4 + 1 \equiv 0$ (wherein $y = \beta Y^2$) to be used instead of (15), as the starting datum. It suffices to write $\alpha = 1$, $X = 1$ in Results (15a), (17), (17a).

11f. *Simple Case* ($\alpha = 1$, $\beta = 2$). This is an important Case. Let there be given

$$N = X^8 + (2Y^2)^4 \equiv 0 \; (\text{mod } p \text{ or } p^\kappa,\ p = 16\varpi + 1;\ XY > 1) \text{...(18)}.$$

with some of the 2^{ic} partitions, or congruences of same form

$$p \text{ or } p^\kappa = a^2 + b^2 = c^2 + 2d^2 = e^2 - 2f^2 = 2f'^2 - e'^2 \text{...(18a)},$$

Then $\qquad 16 Y^8 \equiv -X^8 \pmod{p \text{ or } p^\kappa}$,

and $\qquad (a \pm b)^4 + 4a^4 \equiv 0$, $(a \pm b)^4 + 4b^4 \equiv 0$,

whence $\qquad (a \pm b)^8 = 16a^8$, and $(a \pm b)^8 = 16b^8$.........(19a),

or one of $\qquad c^8 \equiv 16d^8$, $e^8 \equiv 16f^8$, $e'^8 \equiv 16f'^8$.........(19b).

Eliminating 16 gives four roots of $y^8+1\equiv 0$ (mod p or p^κ) in the form

$$y\equiv \pm\frac{Y}{X}\cdot y',\ \text{or}\ \pm\frac{X}{Y}\cdot\frac{1}{y'}\ (\text{mod}\ p\ \text{or}\ p^\kappa)\ldots\ldots (20),$$

where
$$y'\equiv \text{any of}\ y_1,\ \pm 1,\ \text{or}\ y_2\ldots\ \ldots\ldots (20a),$$

and

$$y_1=\text{any of}\ \pm\frac{\mathrm{a}}{\mathrm{b}},\ \pm\frac{\mathrm{b}}{\mathrm{a}};\quad y_2\ \text{any of}\ \pm\frac{\mathrm{c}}{\mathrm{d}},\ \pm\frac{\mathrm{e}}{\mathrm{f}},\ \pm\frac{\mathrm{e}'}{\mathrm{f}'}\ldots (20b).$$

This gives a great choice of formulæ for the terms y_1, y_2 entering into y' and y.

Ex. (of Art. 11*d*).

Given $p=54721=15^4+8^4=15^4+2^4.2^8$; here $\alpha=15$, $\beta=2, X=1$, $Y=2$,

Given also $p=119^2+15.52^2=161^2+2.120^2$ (to eliminate α, β).

Then, by (17), (17*a*), $y=\pm Y.\dfrac{t_1t_2}{u_1u_2}=\pm\dfrac{2.52.161}{119.120}=\pm\dfrac{13.23}{17.15}=\pm\dfrac{299}{17.15}$.

Reducing by Art. 9, $y\equiv\pm\dfrac{299+5p}{17.15}=\pm\dfrac{273904}{17.15}=\pm\dfrac{16112}{15}$ (mod p),

whence $y\equiv\pm\dfrac{16112-2p}{15}\equiv\mp\dfrac{93330}{15}=\mp 6222$, (two 8^{ic} roots).

Or, again, α, β may be eliminated by

$$p=233^2+3.12^2=226^2+5.27^2=161^2+2.120^2.$$

Then, as by (17), (17*a*), $y=\pm Y.\dfrac{t_1t_2t_3}{u_1u_2u_3}=\pm\dfrac{2.12.27.161}{233.226.120}=\pm\dfrac{27.7.23}{10.113.233}$, (or its reciprocal).

Hence $y\equiv\pm\dfrac{263290}{27.7.23}$, which yields on reduction (Art. 9) $y=\pm 18570$ (two 8^{ic} roots).

Ex. (of Art. 11*e*). Given $p=64433=135^2+2.152^2$;

and $1+50^4=1+2^4.5^8\equiv 0$ (mod p).

By (20), (20*b*), $y=\pm\dfrac{1}{Y}\cdot\dfrac{1}{y'}=\pm\dfrac{1}{Y}\cdot\dfrac{\mathrm{d}}{\mathrm{c}}=\pm\dfrac{152}{5.135}=\pm\dfrac{152}{27.25}$.

Reducing by Art. 9, $y=\pm\dfrac{152+6p}{27.25}\equiv\pm\dfrac{386750}{27.25}\equiv\pm\dfrac{15470}{27}$ (mod p).

Also $y\equiv\pm\dfrac{15470-4p}{27}=\mp\dfrac{242262}{27}=\mp\dfrac{26918}{3}$ (mod p),

$\equiv\mp\dfrac{26918-p}{3}=\pm\dfrac{37515}{3}=\pm 12505$ (two 8^{ic} roots).

It will be evident now that the success of the general Method 4° (for finding roots of $y^8+1\equiv 0$) requires that the auxiliary congruences (15*a*, *b*) or (18*a*) which are needed for eliminating α, β, should be either *given*, or else that they should be *easy to form*. The simple form (18) has the advantage that the auxiliary congruences (18*a*) are *always possible*, and that any *one* of them suffices: in this case moreover it is not really necessary to compute an *actual partition* (18*a*) of the modulus (p or p^κ) itself; for the (c, d), (e, f) partitions of the whole number N in (18) can be formed *algebraically* by Art. 4, and will suffice to yield the congruences (19*b*); the partitions of p have, however, the advantage of yielding smaller numbers (c, d), (e, f) than those of N, thus giving easier (subsequent) numerical work.

[Note that this Method 4° (for solving $y^8+1\equiv 0$) is not nearly so general as Method 3° (for solving $y^4+1\equiv 0$), as it is by no means easy to find suitable factorisable numbers (15), (15′), (18), together with the necessary auxiliary congruences (15*a*, *b*), (18*a*). In fact no general Method for solving $y^8+1\equiv 0$ appears to be known, except such as involve the solution of a 2ic congruence (often a difficult matter)].

12. Step ii. *Remaining roots.* When one root (y_1) of either congruence $y^4+1\equiv 0$ or $y^8+1\equiv 0$ (mod p or p^κ) is known then the complete set of four roots of the former, or eight roots of the latter (all $< p$ or p^κ), are given as the *least residues* of y_1^ω (*ω odd*), viz.

$$y_1, y_3, y_5, y_7 \text{ are Residues of } y_1, y_1^3, y_1^5, y_1^7 \text{ for } y^4+1\equiv 0 \ldots (21),$$

$$y_1, y_3, \&c. \ldots y_{15} \text{ are Residues of } y_1, y_1^3, \ldots y_1^{15} \text{ for } y^8+1\equiv 0 \ldots (22).$$

This suggests the following *systematic* mode of computing the roots.

Let $\quad y_2 = \text{least residue of } y_1^2 \text{ (mod } p \text{ or } p^\kappa) \ldots\ldots (23).$

Then the roots may be found in succession, each from the preceding, by multiplying each root, as found, by y_2 and taking the *least residue* of the product,

$$y_3\equiv y_2\cdot y_1;\quad y_5\equiv y_2\cdot y_3;\quad y_7\equiv y_2\cdot y_5;\ \text{and so on} \ldots (24).$$

But, it suffices to compute up to *half the full number* of roots by the above Rule, *i.e.*

Only y_3 for $y^4+1\equiv 0$; Only y_3, y_5, y_7 for $y^8+1\equiv 0 \ldots (25)$;

the remaining roots being given at once by simple subtraction

from the modulus (p or p^κ), since the set of roots can always be arranged in pairs (say y', y'') such that

$$y' + y'' = \text{the modulus } p \text{ or } p^\kappa \text{..............} (26).$$

The roots can also be arranged in pairs whose products satisfy the reciprocal relations,

$$y_1 y_3 \equiv y_5 y_7 \equiv -1;\ y_1 y_7 \equiv y_3 y_5 \equiv +1;\ \text{for } y^4 + 1 \equiv 0 \ldots (27),$$

$$\left.\begin{array}{l} y_1 y_7 \equiv y_3 y_5 \equiv y_9 y_{15} \equiv y_{11} y_{13} \equiv -1 \\ y_1 y_{15} \equiv y_3 y_{13} \equiv y_5 y_{11} \equiv y_7\, y_9 \equiv +1 \end{array}\right\} \text{for } y^8 + 1 \equiv 0 \ldots\ldots (28),$$

in which the law of connexion of the subscripts is obvious.

13. *Tests of work.* When half the full number of roots has been obtained by the above systematic process one of the following Tests of the arithmetical accuracy of the work may be applied to the last root so obtained.

Roots of $y^4 + 1 \equiv 0$. Here y_3 would be the only root so computed.

$y_1 y_3$ should $\equiv -1$..(29a),

$y_2 y_3 \equiv y_5$ should = one of the roots found by subtraction...(29b),

y_3^2 should $\equiv -y_1^2$, or should be a root of $y^2 + 1 \equiv 0$......(29c).

Roots of $y^8 + 1 \equiv 0$. Here y_7 would be the last root computed as above.

$y_1 y_7$ should $\equiv -1$.. ....(29d),

$y_2 y_7 \equiv y_9$ should = one of the roots found by subtraction...(29e),

y_7^2 should $\equiv -y_3^2$, or should be a root of $y^4 + 1 \equiv 0$......(29f),

Any of the above Tests will suffice.

14. *Previous Congruence Tables.* Reuschle's *Tafeln complexer Primzahlen*, Berlin, 1875, gives–on pages 443, 446–short Tables of solutions of the two congruences $y^4 + 1 \equiv 0$, $y^8 + 1 \equiv 0 \pmod p$, extending only to $p \not> 1000$. Only half the full number of roots is given in each case, viz. the roots $< \frac{1}{2}p$. On account of their small extent ($p \not> 1000$), these Tables are of little use for factorisation of high numbers;

and—in consequence of the omission of one half of the roots—are not really convenient even for the search for small divisors (<1000).

15. *Use of Congruence-Tables in Factorisation.* Congruence-Tables, such as described above (Art. 8), give at once the prime divisors (p), and (to a lesser extent) the power-divisors (p^κ), of all numbers $N=(Y^4+1)$, and (Y^8+1), where

$$Y=mp \pm y, \text{ or } =mp^\kappa \pm y \ldots\ldots\ldots\ldots\ldots\ldots (30),$$

and $y=$ any root of $y^4+1\equiv 0$, or $y^8+1\equiv 0 \pmod{p \text{ or } p^\kappa}$..(30$a$).

Such Congruence-Tables are therefore a *most powerful aid* to factorisation of such binomials giving the means (by a comparatively slight examination) of *complete factorisation* of *all* such numbers N and $\frac{1}{2}N$ up to the limits.

N, or $\frac{1}{2}N$, $<p_m^2$, where p_m is the prime next $>$ that for which the Congruence-Table is continuous,

and also of detection of *all* High Primes $P=N$ or $\frac{1}{2}N$ up to the same limit, and also of *complete factorisation* in certain cases up to much higher limits, viz. of all such numbers of form

$$N \text{ or } \tfrac{1}{2}N=(p_1 p_2 \cdots p_r)(p_a^\alpha \cdot p_b^\beta \ldots).P \ldots\ldots\ldots (31),$$

where the primes (p_1, p_2, &c.) and prime-powers (p_a^α, p_b^β, &c.) are among those whose roots (y) are given in the Congruence-Table, and P is a High Prime $<p_m^2$ (as above). The factorisation, and detection of High Primes, can be carried beyond those limits by special means.

[The author's Congruence-Table of $y^4+1\equiv 0$, being complete and continuous (Art. 8) up to $p_m=32441$ has served for the complete factorisation of $N=y^4+1$ up to $y=180$* and of $\frac{1}{2}N=\frac{1}{2}(y^4+1)$ up to $y=249$*, without a break; and also for many higher values of y; and also for the detection of all the High Primes* $\not> 105.10^7$ contained in $N=y^4+1$ up to $y=1000$; see Art. 6b, 7a].

High Factorisations, Ex.

$22946^4+1=41.673.10729.25913.36137$; [18 figures],

$92564^4+1=41.337.457.4057.42073.68113$; [20 figures].

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* The factorisation was specially extended (as stated in Art. 6b) to $y=226$ in case of $N=(y^4+1)$, and to $y=265$ in case of $N=\frac{1}{2}(y^4+1)$, *continuously* (*i.e.* without break), by aid of MS. Tables of solutions of the Congruence $y^2+1\equiv 0$ (mod. p or p^κ), compiled by the author, which are now continuous up to $p \not> 50000$. These Tables thus enabled the detection of High Primes up to 25.10^8.

16. *General Binomial Congruence* ($x > 1$). Solutions (y) of the more general congruences

$$a^4 + y^4 \equiv 0,\ a^8 + y^8 \equiv 0 \ (\text{mod } p \text{ or } p^\kappa);\ [a \text{ constant}] \ldots (32),$$

could be found by various methods similar to those described above (Art. 8 to 13) for the special case when $a = 1$. Also known solutions (Y) of the simple case (where $a = 1$) may be utilised to yield solutions of the more general form by multiplying the simple congruences throughout by a^4 or a^8, whereby at once

$$y = \text{Least Residue of } aY \ (\text{mod } p \text{ or } p^\kappa) \ldots\ldots (32a).$$

When one root has been thus obtained, the other roots may be obtained in the same manner, or by the method of Art. 12.

In the case where $a = 2$, one half of the even roots are given (at sight) as the doubles of the smaller roots (Y) of the simple case when $a = 1$. For since two of the roots (say Y, Y') of $Y^4 + 1 \equiv 0$ and four of the roots (say Y, Y', Y'', Y''') of $Y^8 + 1 \equiv 0$ are always $< \frac{1}{2}p$ or $\frac{1}{2}p^\kappa$, hence

$$y = 2Y,\ 2Y' \text{ are the even roots of } y^4 + 1 = 0 \ldots\ldots\ldots\ldots\ldots (33a),$$

$$y = 2Y,\ 2Y',\ 2Y'',\ 2Y''' \text{ are the even roots of } y^8 + 1 \equiv 0 \ldots (33b),$$

and the remaining (odd) roots are given by subtraction as in (26).

17. *Residuacity and Modularity.* The short notation

$$(q/p)_e = +1, \text{ or } -1, \text{ denotes } q^{\frac{p-1}{e}} \equiv +1, \text{ or } -1 \ (\text{mod } p) \ldots (34),$$

$$\text{where } p = m.e + 1 = \text{prime}, \ [\text{here } e = 2^\kappa] \ldots\ldots\ldots (34a).$$

Here q is said to be a *Residue* or *Non-Residue* of order $e\,(= 2^\kappa)$ of the prime modulus (p); and conversely the prime p is said to be a *Modulus* or *Non-Modulus* of order e of the base q. These properties are styled the *Residuacity* of q, and *Modularity* of p. The question of whether $(q/p) = +1$, or -1, is in general completely determinable, when $e = 2, 4, 8$ (and also $e = 16$ where $q = \pm 2$) by the linear and quadratic relation of q to p, viz.

$$p = m.q + r = \mathrm{a}^2 + \mathrm{b}^2 = \mathrm{c}^2 + 2\mathrm{d}^2 = t^2 \pm qu^2 \ldots\ldots (35).$$

18. *Modularity of Quartans &c., Octavans, &c.* When the modulus is a Quartan, Half-Quartan. &c., *prime*, the rules

for 4^{ic}, 8^{ic}, and sometimes 16^{ic} modularity take simple forms for small bases q (especially for prime bases q). Thus

$$y = 2mq \text{ gives } (q/p)_8 = +1,$$
$$\text{for } p = (x^4 + y^4) \text{ and } (x^8 + y^8)\,;\ [q \text{ prime}] \ldots\ldots (36),$$
$$x \mp y = 2mq \text{ or } 4mq \text{ gives } (q/p)_8 = +1,$$
$$\text{for } p = \tfrac{1}{2}(x^4 + y^4) \text{ and } \tfrac{1}{2}(x^8 + y^8) \ldots\ldots (37),$$

appear to be in general sufficient (though by no means always necessary) conditions for $(q/p)_8 = +1$, when q is an odd prime: when q is composite, the conditions are more complex. The detailed criteria of 2^{ic}, 4^{ic}, 8^{ic} residuacity of the small bases $\pm q = 2, 3, 5, \ldots, 12$ for such moduli are given in the four Tables A, B, C, D, following:

Criteria.	*Tab.* *Mod.*	*Tab.* *Mod.*
2^{ic}, 4^{ic}, 8^{ic},	A; $p = x^4 + y^4$	B; $p = \frac{1}{2}(x^4 + y^4)$
2^{ic}, 4^{ic}, 8^{ic}, 16^{ic},	C; $p = x^8 + y^8$	D; $p = \frac{1}{2}(x^8 + y^8)$

The same criterion applies to both $\pm q$ alike throughout these Tables, except in the two right-hand columns of Tab. B, D, wherein the criteria apply to only one of $\pm q$, viz. to that case which *gives the simplest criterion*, viz.

For $+q$ or $-q$, according as q, $\frac{1}{2}q$, $\frac{1}{4}q$, &c. $= (4k+1)$ or $(4k-1)$.

Tab. B; $p = \frac{1}{2}(x^4 + y^4) = 8\varpi + 1$; $(q/p)_8$ given for $q = +2, 5, 10$; $\bar{3}, \bar{6}, \bar{7}, \overline{11}, \overline{12}$;

Tab. D; $p = \frac{1}{2}(x^8 + y^8) = 16\varpi + 1$; $(q/p)_{16}$ given for $q = +2, 5, 10$; $\bar{3}, \bar{6}, \bar{7}, \overline{11}, \overline{12}$;

The cases of $\pm q$ are connected by the simple relations

$$p = \tfrac{1}{2}(x^4 + y^4) = 8\varpi + 1\,; \quad (q/p)_8 \cdot (\bar{q}/p)_8 = (-1)^{\varpi} \ldots\ldots\ldots\ldots (38a),$$
$$p = \tfrac{1}{2}(x^8 + y^8) = 16\varpi + 1\,; \quad (q/p)_{16} \cdot (\bar{q}/p)_{16} = (-1)^{\varpi} \ldots\ldots\ldots\ldots (38b).$$

These criteria have been reduced* by the author from the general criteria of 4^{ic} and 8^{ic} modularity for the small bases $(\pm q)$ stated.

[The signs * † ‡ § ¶ in Tables B, D denote *repetition* of the condition so marked in the column to left in which the sign first occurs. The sign, &c., indicates that an additional condition (not easily included in the Table) is needed. In the case of $q = \overline{11}$, the upper signs are to be used throughout, or the lower signs throughout, each line].

19. 16^{ic} *and* 32^{ic} *Modularity.* The criteria hitherto discovered extend only up to order 8 in general. In all cases

$$(q/p)_8 = +1 \text{ involves } (q/p)_{16} = \pm 1, \ [\text{when } p = 16\varpi + 1] \ldots\ldots\ldots (39).$$

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* And have also been tested on all quartan and half-quartan primes $\not> 100000$, (and in a few cases to a higher limit).

q	$(q/p)_2=-1$	$(q/p)_2=+1$	$(q/p)_4=-1$	$(q/p)_4=+1$	$(q/p)_8=-1$	$(q/p)_8=+1$
± 2	·	$y=\varepsilon$	$y=2\omega$	$y=2\varepsilon$	·	$y=4i$
± 3	$xy\mp 3\varepsilon$	$xy=3\varepsilon$	$x=3\omega$	$y=3\varepsilon$	·	$y=6i$
± 5	$xy\mp 5\varepsilon$	$xy=5\varepsilon$	$x=5\omega$	$y=5\varepsilon$	·	$y=10i$
± 6	$xy\mp 6i$	$xy=6i$	i. $y=6\omega$ ii. $x=3\omega,\ y=4i$	$y=6\varepsilon$ $x=3\omega,\ y=2\omega$	· ·	$y=12i$ $x=3\omega,\ y=2\omega$
± 7	i. $xy\mp 7\varepsilon$ ii. $x\mp y\mp 7\omega$	$xy=7\varepsilon$ $x\mp y=7\omega$	· $x\mp y=7\omega$	$xy=7\varepsilon$ ·	$x=7\omega$ ·	$y=7\varepsilon$ ·
± 10	$xy\mp 10i$	$xy=10i$	i. $y=10\omega$ ii. $x=5\omega,\ y=4i$	$y=10\varepsilon$ $x=5\omega,\ y=2\omega$	· $x=5\omega,\ y=2\omega$	$y=20i$ ·
± 11	i. $xy\mp 11\varepsilon$ ii. $3x\mp y\mp 11\omega$ iii. $x\mp 3y\mp 11\omega$	$xy=11\varepsilon$ $3x\mp y=11\omega$ $x\mp 3y=11\omega$	$x=11\omega$ $3x\mp y=11\omega$ ·	$y=11\varepsilon$ · $x\mp 3y=11\omega$	· · $x\mp 3y=11\omega$	$y=11\varepsilon$ · ·
± 12	$xy\mp 6i$	$xy=6i$	$x=3\omega$	$y=6i$	$y=6\omega$	$y=12i$

Criteria of $(q/p)_e=\pm 1$; $[e=2, 4, 8]$. $p=\frac{1}{2}(x^4+y^4)=8\varpi+1$; $[x=\omega,\ y=\omega]$. TAB. B.

	$(q/p)_2=-1$	$(q/p)_2=+1$	$(q/p)_4=-1$	$(q/p)_4=+1$	q	$(q/p)_8=-1$	$(q/p)_8=+1$
± 2	·	$x=\omega,\ y=\omega$	$x\mp y=4\omega$	$x\mp y=4\varepsilon$	2	$x\mp y=8\omega$	$x\mp y=8\varepsilon$
± 3	$xy=3\omega$	$x\mp y=3\varepsilon$	·	$x\mp y=3\varepsilon$	$\bar{3}$	$x\mp y=6\omega$	$x\mp y=6\varepsilon$
± 5	$xy=5\omega$	$xy\mp 5\omega$	$x\mp 2y=5\omega$	$x\mp y=5\varepsilon$	5	$x\mp y=10\omega$	$x\mp y=10\varepsilon$
± 6	$xy=3i$	$x\mp y=3\varepsilon$	i. $x\mp y=12\omega$ & 2ω ii. $x\mp y=4\omega$ & 6ω	$x\mp y=12e$ & 2ω $x\mp y=4\varepsilon$ & 6ω	$\bar{6}$	i. $x\mp y=24\omega$ & 2ω ii. $x\mp y=8\varepsilon$ & 6ω	$x\mp y=24\varepsilon$ & 2ω $x\mp y=8\omega$ & 6ω
± 7	i. $xy\mp 7\omega$ ii. $x\mp y\mp 7\varepsilon$	$xy=7\omega$ $x\mp y=7\varepsilon$	$xy=7\omega$ ·	· $x\mp y=7\varepsilon$	$\bar{7}$	· ·	· $x\mp y=7\varepsilon$
± 10	$xy=5i$	$x\mp y=10i$ $*x\mp 2y=5\omega$	i. $x\mp y=20\omega$ & 2ω ii. $x\mp y=4\omega$ & 10ω * & $x\mp y=4\varepsilon$	$x\mp y=20\varepsilon$ & 2ω $x\mp y=4\varepsilon$ & 10ω * & $x\mp y=4\omega$†	10	$x\mp y=40\omega$ & 2ω $x\mp y=8\varepsilon$ & 10ω * & † &c.	$x\mp y=40\varepsilon$ & 2ω $x\mp y=8\omega$ & 10ω * & † &c.
± 11	i. $x\mp y\mp 11\varepsilon$ ii. $x\mp 2y\mp 11\omega$ iii. $2x\mp y\mp 11\omega$	$x\mp y=11\varepsilon$ $x\mp 2y=11\omega$ $2x\mp y=11\omega$	· · ·	† $x\mp y=11\varepsilon$ ‡ $x\mp 2y=11\omega$ § $2x\mp y=11\omega$	$\overline{11}$	† & $xy=4i\mp 1$ ‡ & $xy=4i\pm 1$ § & $xy=4i\pm 1$	† & $xy=4i\pm 1$ ‡ & $xy=4i\mp 1$ § & $xy=4i\mp 1$
± 12	$xy=3i$	$x\mp y=6i$	·	$x\mp y=6i$	$\overline{12}$	i. $x\mp y=12\omega$ ii. $x\mp y=4\varepsilon$ & 6ω	$x\mp y=12\varepsilon$ $x\mp y=4\omega$ & 6ω

Criteria of $(q/p)_e = \mp 1$; $[e = 2, 4, 8, 16]$. $p = x^8 + y^8 = 32\varpi + 1$, $[x = \omega,\ y = \varepsilon]$. TAB. C.

q	$(q/p)_2 = -1$	$(q/p)_2 = +1$	$(q/p)_4 = -1$	$(q/p)_4 = +1$	$(q/p)_8 = -1$	$(q/p)_8 = +1$	$(q/p)_{16} = -1$	$(q/p)_{16} = +1$
± 2	.	$y = \varepsilon$	.	$y = \varepsilon$	.	$y = \varepsilon$	.	$y = \varepsilon$
± 3	$xy \neq 3\varepsilon$	$xy = 3\varepsilon$	$x = 3\omega$	$y = 3\varepsilon$	.	$y = 3\varepsilon$	.	$y = 3\varepsilon$
± 5	$xy \neq 5\varepsilon$	$xy = 5\varepsilon$	$x = 5\omega$	$y = 5\varepsilon$	.	$y = 5\varepsilon$	.	$y = 5\varepsilon$
± 6	$xy \neq 6i$	$xy = 6i$	$x = 3\omega,\ y = \varepsilon$	$y = 6i$	.	$y = 6i$	.	$y = 6i$
± 7	$xy \neq 7\varepsilon$	$xy = 7\varepsilon$	.	$xy = 7\varepsilon$	$x = 7\omega$	$y = 7\varepsilon$	.	$y = 7\varepsilon$
± 7	$x \mp y \neq 7\omega$	$x \mp y = 7\omega$	$x \mp y = 7\omega$	.	.	.	.	.
± 10	$xy \neq 10i$	$xy = 10i$	$x = 5\omega,\ y = \varepsilon$	$y = 10i$	.	$y = 10i$	.	$y = 10i$
± 11	$xy \neq 11\varepsilon$	$xy = 11\varepsilon$	$x = 11\omega$	$y = 11\varepsilon$	.	$y = 11\varepsilon$	.	$y = 11\varepsilon$
± 11	$x \mp 2y \neq 11\omega$	$x \mp 2y = 11\omega$	$x \mp 2y = 11\omega$	.	.	.	.	.
± 11	$2x \mp y \neq 11\varepsilon$	$2x \mp y = 11\varepsilon$	.	$2x \mp y = 11\varepsilon$	$2x \mp y = 11\varepsilon$	.	.	.
± 12	$xy \neq 6i$	$xy = 6i$	$x = 3\omega$	$y = 6i$	.	$y = 6i$	.	$y = 6i$

Criteria of $(q/p)_e = \pm 1$; $[e = 2, 4, 8, 16]$. $p = \frac{1}{2}(x^8 + y^8) = 16\varpi + 1$; $[x = \omega,\ y = \omega]$. TAB. D.

q	$(q/p)_2 = -1$	$(q/p)_2 = +1$	$(q/p)_4 = -1$	$(q/p)_4 = +1$	$(q/p)_8 = -1$	$(q/p)_8 = +1$	q	$(q/p)_{16} = -1$	$(q/p)_{16} = +1$
± 2	.	$x = \omega,\ y = \omega$	.	$x = \omega,\ y = \omega$	$x \mp y = 4\omega$ & 2ω	$x \mp y = 4\varepsilon$	2	$x \mp y = 8\omega$	$x \mp y = 8\varepsilon$
± 3	$xy = 3\omega$	$x \mp y = 3\varepsilon$	.	$x \mp y = 3\varepsilon$	.	$x \mp y = 3\varepsilon$	$\bar{3}$	$x \mp y = 6\omega$	$x \mp y = 6\varepsilon$
± 5	$xy = 5\omega$	$xy \neq 5\omega$	.	$xy \neq 5\omega$	$x \mp 2y = 5\omega$	$x \mp y = 5\varepsilon$	5	$x \mp y = 10\omega$	$x \mp y = 10\varepsilon$
± 6	$xy = 3\omega$	$x \mp y = 6i$	.	$x \mp y = 6i$	$x \mp y = 12\omega$ & 2ω $x \mp y = 4\omega$ & 6ω	$x \mp y = 12\varepsilon$ & 2ω $x \mp y = 4\varepsilon$ & 6ω	$\bar{6}$	i. $x \mp y = 24\omega$ & 2ω ii. $x \mp y = 8\varepsilon$ & 6ω	$x \mp y = 24\varepsilon$ & 2ω $x \mp y = 8\omega$ & 6ω
± 7	$xy \neq 7\omega$	$xy = 7\omega$	$xy = 7\omega$	.	.	.	$\bar{7}$	i. .	.
± 7	$x \mp y \neq 7\varepsilon$	$x \mp y = 7\varepsilon$	.	$x \mp y = 7\varepsilon$	.	$x \mp y = 7\varepsilon$		ii. .	$x \mp y = 7\varepsilon$
± 10	$xy = 5\omega$	$x \mp y = 5\varepsilon$ * $x \mp 2y = 5\omega$	.	$x \mp y = 20i$ & 2ω $x \mp y = 4i$ & 10ω $x \mp y = 4i$ & *	$x \mp y = 20\omega$ & 2ω $x \mp y = 4\omega$ & 10ω $x \mp y = 4\varepsilon$ & *	$x \mp y = 20\varepsilon$ & 2ω $x \mp y = 4\varepsilon$ & 10ω † $x \mp y = 4\omega$ & *	10	i. $x \mp y = 40\omega$ & 2ω ii. $x \mp y = 8\varepsilon$ & 10ω iii. *, & † &c.	$x \mp y = 40\varepsilon$ & 2ω $x \mp y = 8\omega$ & 10ω * & † &c.
± 11	$x \mp y \neq 11\varepsilon$	$x \mp y = 11\varepsilon$	.	$x \mp y = 11\varepsilon$	.	‡ $x \mp y = 11\varepsilon$	$\overline{11}$	‡ & $xy = 4i \mp 1$	‡ & $xy = 4i \pm 1$
± 11	$x \mp 3y \neq 11\varepsilon$	$x \mp 3y = 11\varepsilon$	.	$x \mp 3y = 11\varepsilon$	.	§ $x \mp 3y = 11\varepsilon$		§ & $xy = 4i \pm 1$	§ & $xy = 4i \mp 1$
± 11	$3x \mp y \neq 11\varepsilon$	$3x \mp y = 11\varepsilon$	.	$3x \mp y = 11\varepsilon$	.	¶ $3x \mp y = 11\varepsilon$		¶ & $xy = 4i \pm 1$	¶ & $xy = 4i \mp 1$
± 12	$xy = 3i$	$x \mp y = 6i$	.	$x \mp y = 6i$	.	$x \mp y = 6i$	$\overline{12}$	i. $x \mp y = 12\omega$ ii. $x \mp y = 4\varepsilon$ & 6ω	$x \mp y = 12\varepsilon$ $x \mp y = 4\omega$ & 6ω

No criteria are as yet known for $(q/p)_{16}$, except in the case when $q = \pm 2$, for which the criteria* are shown in the following Table, $[p = 16\varpi + 1$ throughout$]$:—

mod p;	$(2/p)_{16} = +1$	mod p;	$(2/p)_{16} = -1$;	$(2/p)_{16} = +1$ ....
(x^4+y^4);	$y = 2\varepsilon$	$\frac{1}{2}(x^4+y^4)$;	$x \mp y = 16\omega$;	$x \mp y = 16\varepsilon$....(40a);
(x^8+y^8);	$y = \varepsilon$	$\frac{1}{2}(x^8+y^8)$;	$x \mp y = 8\omega$;	$x \mp y = 8\varepsilon$......(40b).

These criteria are for $q = +2$; those for $q = \pm 2$ are connected by the relations (38a, b).

Also in all cases $(2/p)_{16} = +1$, involves $(2/p)_{32} = \mp 1$ (when $p = 32\varpi + 1$); but no criteria are known for the sign of $(2/p)_{32}$. Thus the known criteria for $(q/p)_e = \mp 1$, where $e = 2^\kappa$, stop at $e = 16$ for the case of $y = \pm 2$, and at $e = 8$ when $q > 2$.

20. *New Criteria for* $(2/p)_{32}$ *and* $(q/p)_{16}$. It seems probable that the forms $p = (x^8 + y^8)$ and $\frac{1}{2}(x^8 + y^8)$ bear *much the same relations* to both $(2/p)_{32}$ and $(q/p)_{16}$ that $p = (x^4 + y^4)$ and $\frac{1}{2}(x^4 + y^4)$ are known to bear towards both $(2/p)_{16}$ and $(q/p)_8$. This seems to involve as criteria, in many cases sufficient, (but not always necessary)—

mod p	$(2/p)_{32} = -1$;	$(2/p)_{32} = +1$	$(q/p)_{16} = -1$;	$(q/p)_{16} = +1$ ....
(x^8+y^8)	$y = 2\omega$;	$= 2\varepsilon$	. ;	$y = \varepsilon q$........(41b),
$\frac{1}{2}(x^8+y^8)$	$x \mp y = 16\omega$;	$x \mp y = 16\varepsilon$	$x \mp y = 2\omega q$; (in some cases)	$x \mp y = 2\varepsilon q$ ...(41b),

or, more generally thus:—

if $\qquad p = x^4 + y^4, \qquad$ and $P = x^8 + y^8$, [same x, y],

$\qquad p' = \frac{1}{2}(x^4 + y^4)$, and $P' = x^8 + y^8$, [same x, y],

then (when $q \not> 12$) the criteria of modularity of (p, P), (p', P') have—(with some slight modification found necessary by induction)—the *same form* in x, y in the following pairs

$$(2/p)_{16} = \mp 1 \ \& \ (2/P)_{32} = \mp 1; \ (2/p')_{16} = \mp 1 \ \& \ (2/P')_{32} = \mp 1 \ldots (42a,b),$$

$$(q/p)_8 = \mp 1 \ \& \ (q/P)_{16} = \mp 1; \ (q/p')_8 = \mp 1 \ \& \ (q/P')_{16} = \mp 1 \ldots (42c,d).$$

The two right-hand columns of Tables C, D preceding, have been drawn up in accordance with these assumed rules:

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* These are due to the author: see his Paper *On 2 as a* 16^{ic} *Residue* in *Proc. Lond. Math. Soc.*, Vol. XXVII., 1895, p. 85 *et. seq.*

these criteria are of course somewhat conjectural, as they depend only on formal analogy and on a small quantity of numerical evidence (given below).

20*a*. *Numerical evidence.* There are so few primes practically available of the forms $p=(x^8+y^8)$ and $\frac{1}{2}(x^8+y^8)$ that very little numerical testing can be tried. *All* the data at present) available are shown in the accompanying Table E. The body of the Table shews the residue-index ($e=2, 4, 8, 16, 32$, &c.) actually found (by computation) to give $(q/p)_e=-1$, or $+1$, as shown in the head-line: the column headed $(p-1)/\Omega$ shows the *highest power* of 2 contained in $(p-1)$.

TAB. E.

p	p	x, y	$\frac{p-1}{\Omega}$	$q=$ $(q/p)_e=$	2 $1, \bar{1}$	-3 $1, \bar{1}$	5 $1, \bar{1}$	-6 $1, \bar{1}$	-7 $1, \bar{1}$	10 $1, \bar{1}$	-11 $1, \bar{1}$	-12 $1, \bar{1}$
(x^8+y^8)	257	1, 2	256	$e=$	16, 32	. 2	. 2	. 2	. 2	. 2	2, 4	. 2
	65537	1, 4	2^{16}	$e=$	$2^{11}, 2^{12}$	. 2	. 2	. 2	. 2	. 2	. 2	. 2
	2070241	5, 6	32	$e=$	16, 32	16, 32	2, 4	32, .	. 2	2, 4	. 2	16, 32
	100006561	3, 10	32	$e=$	16, 32	2, 4	32, .	2, 4	2, 4	16, 32	. 2	2, 4
$\frac{1}{2}(x^8+y^8)$	198593	3, 5	64	$e=$	8, 16	. 2	. 2	. 2	. 2	. 2	. 2	. 2
	21523361	1, 9	32	$e=$	8, 16	. 2	8, 16	. 2	. 2	16, 32	. 2	. 2
	107182721	3, 11	128	$e=$	8, 16	. 2	4, 8	. 2	32, 64	4, 8	. 2	. 2
	407865361	1, 13	16	$e=$	4, 8	16, .	4, 8	4, 8	16, .	16, .	. 2	8, 16

21. *Residuacity of* 2 *to mod* (x^4+y^4), *and* (x^2+y^2), *&c.* It is worth noting that:—

If $p=x^4+y^4$ and $p'=x^2+y^2$ be *prime* (with same x, y),

then $(2/p)_{16}=+1$, $(2/p')_2=+1$ involve one another.........(43*a*).

Also,

if $p=\frac{1}{2}(x^4+y^4)$ and $p'=\frac{1}{2}(x^2+y^2)$ be *prime* (with same x, y),

then $(2/p)_{16}=+1$, $(2/p')_8=+1$ involve one another (if $p'=16\varpi'+1$)...(43*b*).

22. 32^{ic} *Residuacity of* 2 *with Quartans and Half-Quartans.* The 16^{ic} criteria of the base 2 with respect to Quartan and Half-Quartan primes are so extremely simple (Art. 19), that it seems probable that the 32^{ic} criteria *with such primes* should be much simpler than with primes in general, and therefore (in absence of any direct theory) more easily discoverable by numerical trial. As a step towards discovering such a criterion, the author has computed the actual (∓ 1) value* of $(2/p)_{32}$ for *all* primes p,

$p=x^4+y^4=32\varpi+1$, $\not> 9$ million, and some higher,

$p=\frac{1}{2}(x^4+y^4)=32\varpi+1$, $\not> 9$ million, and some higher.

The results are shown in Table VIII, which is divided (down the middle) into two parts:—

The left Table shows those primes for which $(2/p)_{32}=-1$,

The right Table shows those primes for which $(2/p)_{32}=+1$.

The column E shows the *highest power* of 2 in $(p-1)$, and the column e (on the right) shows the *highest power* of 2 in the residue-index, *i.e.* such that $(2/p)_e=+1$.

32. *Tables of primes* (Tab. I. to VII). These Tables, immediately following, are explained in Art. 7, 7*a*.

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* The author has failed as yet in deducing any definite criterion from these results: but it seems worth while placing them on record (as they are the outcome of heavy work) for future use.

Quartan Primes, $p=(x^4+y^4)$, [x *odd*, y *even*]. TAB. I.

p	x, y	p	x, y	p	x, y	p	x, y
17	1, 2	149057	17, 16	824641	11, 30	2070241	25, 36
97	3, 2	151057	19, 12	838561	13, 30	2085217	3, 38
257	1, 4	160001	1, 20	847601	25, 26	2168657	17, 38
337	3, 4	160081	3, 20	867281	29, 20	2279617	21, 38
641	5, 2	166561	9, 20	893521	17, 30	2351857	39, 14
881	5, 4	168737	19, 14	941537	29, 22	2378977	39, 16
1297	1, 6	204481	21, 10	944257	31, 12	2473441	39, 20
2417	7, 2	243521	17, 20	961937	31, 14	2522257	33, 34
2657	7, 4	260017	21, 16	988417	27, 26	2566561	9, 40
3697	7, 6	279857	23, 2	1049201	5, 32	2616577	27, 38
4177	3, 8	280097	23, 4	1050977	7, 32	2684161	37, 30
4721	5, 8	283937	23, 8	1055137	9, 32	2690321	19, 40
6577	9, 2	284881	15, 22	1089841	23, 30	2754481	21, 40
10657	9, 8	289841	23, 10	1146097	27, 28	2825777	41, 2
12401	7, 10	317777	17, 22	1178897	19, 32	2836961	35, 34
14657	11, 2	331777	1, 24	1224337	33, 14	2839841	23, 40
14897	11, 4	334177	7, 24	1328417	23, 32	2922737	37, 32
15937	11, 6	346417	11, 24	1336337	1, 34	2930737	41, 18
16561	9, 10	360337	13, 24	1336417	3, 34	3112321	5, 42
28817	13, 4	384817	23, 18	1336961	5, 34	3157537	41, 24
38561	13, 10	391921	25, 6	1338737	7, 34	3195217	17, 42
39041	5, 14	394721	25, 8	1342897	9, 34	3242017	19, 42
49297	13, 12	411361	25, 12	1345921	33, 20	3362017	39, 32
54721	15, 8	457057	3, 26	1350977	11, 34	3391537	23, 42
65537	1, 16	459377	7, 26	1364897	13, 34	3428801	43, 10
65617	3, 16	462097	19, 24	1466657	19, 34	3439537	43, 12
66161	5, 16	463537	9, 26	1501921	35, 6	3457217	43, 14
66977	13, 14	471617	11, 26	1521361	35, 12	3553777	37, 36
80177	11, 16	531457	27, 2	1682017	7, 36	3578801	43, 20
83537	17, 2	587297	19, 26	1763137	17, 36	3635761	41, 30
83777	17, 4	596977	27, 16	1800577	33. 28	3649777	39, 34
89041	15, 14	614657	1, 28	1809937	19, 36	3653057	43, 22
105601	5, 18	621217	9, 28	1874177	37, 2	3750577	43, 24
107377	7, 18	643217	13, 28	1874417	37, 4	3818977	29, 42
119617	11, 18	728017	29, 12	1878257	37, 8	3874337	41, 32
121937	17, 14	736817	23, 26	1912577	37, 14	3942577	21, 44
130337	19, 2	744977	19, 28	1959457	23, 36	3959297	37, 38
131617	19, 6	745697	29, 14	1972097	31, 32	4035217	31, 42
134417	19, 8	812257	29, 18	2034161	37, 20	4100641	45, 2
140321	19, 10	812401	7, 30	2043617	29, 34	4100881	45, 4

Tab. I. (*continued*).

Quartan Primes, $p=x^4+y^4$, [x *odd*, y even].

p	x, y	p	x, y
4104721	45, 8	7435921	33, 50
4162097	41, 34	7439681	47, 40
4279537	27, 44	7506097	21, 52
4398577	39, 38	7591457	23, 52
4467377	43, 32	7813777	51, 32
4477457	1, 46	7843057	27, 52
4477537	3, 46	7891777	53, 6
4478081	5, 46	7894577	53, 8
4505377	41, 36	7900481	53, 10
4506017	13, 46	7911217	53, 12
4560977	17, 46	7928897	53, 14
4607777	19, 46	8050481	53, 20
4671937	21, 46	8124461	37, 50
4715281	45, 28	8222257	53, 24
4755137	43, 34	8324801	49, 40
4879937	47, 4	8503057	1, 54
4880977	47, 6	8503681	5, 54
4910897	41, 38	8505137	53, 28
4918097	47, 14	8531617	13, 54
5039681	47, 20	8586577	17, 54
5211457	47, 24	8627777	47, 44
5308417	1, 48	8633377	19, 54
5309041	5, 48	8812241	35, 52
5385761	41, 40	8939057	53, 32
5391937	17, 48		
5436961	45, 34		
5663377	33, 46		
5764817	49, 2		
5768897	49, 8		
5785537	49, 12		
5978801	43, 40		
6015697	29, 48		
6185761	45, 38		
6252401	7, 50		
6278561	13, 50		
6333521	17, 50		
6444481	21, 50		
6765217	51, 2		
6769297	51, 8		
6775201	51, 10		
6790897	39, 46		
6925201	51, 20		
6964817	47, 38		
6999457	51, 22		
7101137	49, 34		
7166897	43, 44		
7222177	51, 26		
7326257	11, 52		

Tab. III.

High Quartan Primes.

$p=(x^4+y^4)$, [x *odd*, y *even*].

	p	x, y
C	9834497	1, 56
	29986577	1, 74
B	40960001	1, 80
	45212177	1, 82
	59969537	1, 88
B	65610001	1, 90
Da	100000081	3, 100
	100006561	9, 100
	126247697	1, 106
	193877777	1, 118
	303595777	1, 132
	384160001	1, 140
	406586897	1, 142
	562448657	1, 154
	655360001	1, 160
	723394817	1, 164
	916636177	1, 174
	1049760001	1, 180
	1416468497	1, 194
	1536953617	1, 198
	1731891457	1, 204
	1944810001	1, 210

Tab. IV.

High Half-Quartan Primes.

$p=\frac{1}{2}(1+y)^4$, [y *odd*].

	p	x, y
B	12705841	1, 71
B	14199121	1, 73
BJ	21523361	1, 81
	56275441	1, 103
	60775313	1, 105
	81523681	1, 113
	87450313	1, 115
	100266961	1, 119
	138461441	1, 129
	273990641	1, 153
	370600313	1, 165
	407865361	1, 169
	427518041	1, 171
	784119601	1, 199
	849090841	1, 203
	883050313	1, 205
	1984563001	1, 251
	2249930281	1, 259

Half-Quartan Primes, $p = \frac{1}{2}(x^4 + y^4)$, [$x$ & y *odd*]. TAB. II.

p	x, y	p	x, y	p	x, y	p	x, y
1	1, 1	353641	29, 1	1975121	43, 27	4591801	49, 43
41	3, 1	353681	29, 3	2005841	41, 33	4617073	55, 17
313	5, 1	378953	29, 15	2057633	45, 11	4672553	55, 21
353	5, 3	405641	27, 23	2092073	45, 17	4715233	55, 23
1201	7, 1	450881	29, 21	2093801	39, 37	4795481	51, 41
3593	9, 5	461801	31, 3	2163193	41, 35	4928953	55, 29
4481	9, 7	462073	31, 5	2171161	43, 31	4932713	49, 45
7321	11, 1	465041	31, 9	2190233	45, 23	5101961	53, 39
8521	11, 7	476041	31, 13	2439881	47, 3	5278001	57, 1
10601	11, 9	487073	31, 15	2440153	47, 5	5319761	57, 17
14281	13, 1	548953	29, 25	2441041	47, 7	5473313	57, 25
14321	13, 3	559001	31, 21	2447161	47, 11	5654641	53, 43
14593	13, 5	593273	33, 5	2454121	47, 13	5822441	51, 47
21601	13, 11	594161	33, 7	2481601	47, 17	5988193	55, 41
26513	15, 7	750313	35, 1	2537081	47, 21	6028313	57, 35
32633	15, 11	750353	35, 3	2705561	47, 27	6058993	59, 5
41761	17, 1	757633	35, 11	2793481	47, 29	6083993	59, 15
41801	17, 3	764593	35, 13	2866121	43, 39	6123841	59, 19
42073	17, 5	792073	35, 17	2882441	49, 3	6198601	59, 23
42961	17, 7	815401	31, 29	2901601	47, 31	6253993	59, 25
49081	17, 11	937121	37, 3	2907713	49, 15	6265001	51, 49
56041	17, 13	940361	37, 9	2947561	49, 19	6324401	59, 27
66361	19, 7	951361	37, 13	3032801	47, 33	6412321	59, 29
67073	17, 15	1002241	37, 19	3122281	43, 41	6520441	59, 31
72481	19, 11	1016033	35, 27	3148121	49, 27	6690881	57, 41
90473	19, 15	1054721	33, 31	3190153	47, 35	6922921	61, 1
97241	21, 1	1132393	37, 25	3236041	49, 29	6930241	61, 11
97553	21, 5	1156721	39, 1	3344161	49, 31	6948233	61, 15
104561	21, 11	1157033	39, 5	3383801	51, 7	6995761	59, 37
106921	19, 17	1198481	39, 17	3522521	51, 23	7020161	61, 21
111521	21, 13	1398841	37, 31	3577913	51, 25	7215401	59, 39
139921	23, 1	1414081	41, 7	3736241	51, 29	7471561	59, 41
141121	23, 7	1416161	41, 9	3759713	45, 43	7768081	59, 43
165233	23, 15	1420201	41, 11	3819481	49, 37	7941641	63, 19
195353	25, 3	1510121	41, 21	3948521	53, 9	8160401	57, 49
198593	25, 9	1510361	39, 29	3952561	53, 11	8230121	63, 29
205081	23, 19	1618481	39, 31	3987001	53, 17	8338241	63, 31
237073	25, 17	1678601	41, 27	4132913	51, 35	8925313	65, 1
237161	23, 21	1687393	37, 35	4295281	49, 41	8928593	65, 9
266921	27, 7	1709713	43, 5	4298881	53, 29	8967073	65, 17
280001	27, 13	1710601	43, 7	4319681	51, 37		
307481	27, 17	1734713	43, 15	4589593	55, 13		

TAB. V.

High Primes $p=\frac{1}{\mu}.(1+y^4)$, [*y even*]; $\mu=17, 41, 73, 89, 97$, &c.

p	y,	μ	p	y,	μ	p	y,	μ
9167489	712	μ	36268129	354	μ	165991393	468	μ
9793969	380	μ	36269593	950	μ	194213177	564	μ
10088489	934	μ	39818929	392	μ	201796057	626	μ
10677089	3266	μ	40054897	356	μ	205048201	904	μ
11165137	528	μ	40514561	162	17	206063593	368	89
11505017	942	μ	44669593	736	μ	210378169	14506	μ
11966641	252	μ	44711201	548	μ	210469913	14506	μ
12321041	474	μ	49916473	378	μ	238275601	994	μ
13294121	502	μ	50855561	486	μ	241632361	740	μ
13374089	336	μ	50897897	330	μ	273148633	890	μ
14394409	362	μ	51244313	572	μ	287803777	834	μ
14579681	970	μ	51483121	172	17	L 308761441	8192	μ
14641849	456	μ	52216841	608	μ	334140193	762	μ
15120673	766	μ	57734881	676	μ	361562353	280	17
L 15790321	128	17	60880681	872	μ	389961553	830	μ
16673401	674	μ	63798737	668	μ	400495049	802	μ
16782449	326	μ	65798849	700	μ	431830177	470	μ
16898729	684	μ	73805233	680	μ	463504289	636	μ
17137129	514	μ	73853993	644	μ	463891201	298	17
17957969	956	μ	74046641	666	μ	535609489	496	μ
18145313	388	μ	74524553	728	μ	563676649	602	μ
18463497	434	μ	75297473	724	μ	599786777	396	41
19050289	750	μ	77938409	586	μ	606454393	482	89
20260553	402	μ	78374441	816	μ	611416873	870	μ
20361377	458	μ	82509577	814	μ	613350137	460	73
20905193	726	μ	86631049	282	73	613775969	718	μ
21333761	138	17	94106561	422	μ	670464121	744	μ
L 22253377	4096	μ	97089257	686	μ	707646281	558	μ
22925033	806	μ	97905289	630	μ	714666481	332	17
24132457	416	μ	98672257	446	μ	775275233	688	μ
24290249	398	μ	Lf 99990001	1000	μ	788278297	424	41
25068521	960	μ	113607841	324	97	825799841	532	97
25397761	770	μ	113947529	302	73	937534777	836	μ
25737017	464	μ	118821361	212	17			
25744921	366	μ	126041329	908	μ			
27126929	862	μ	126431801	976	μ			
27475081	372	μ	134472673	444	μ			
29497513	544	μ	137123009	534	μ			
31142473	756	μ	141456017	722	μ			
31582673	592	μ	157341673	344	89			

TAB. VI.

High Primes, $p=\frac{1}{\mu}\cdot\frac{1}{2}(1+y^4)$, [$y$ *odd*]; $\mu=17, 41, 73, 89, 97,$ &c.

p	y, μ	p	y, μ	p	y, μ
9037817	959 μ	31308961	779 μ	186643993	825 μ
9085337	283 μ	33510401	295 μ	210907993	291 17
9226673	699 μ	33621673	813 μ	219192097	737 μ
9485321	167 41	34040569	279 89	233726369	937 μ
9661777	547 μ	37529113	189 17	240110729	887 μ
9946609	627 μ	39606769	767 μ	268083401	727 μ
10316017	197 73	39785017	369 μ	288959497	967 μ
10509841	303 μ	40124537	755 μ	319585921	651 μ
10771417	347 μ	41912953	877 μ	334629161	407 41
11616697	433 μ	B 42521761	243 41	436337753	349 17
11756681	831 μ	42526489	195 17	468260633	549 97
12004217	215 89	43026433	585 μ	468571633	899 μ
12452641	797 μ	43068329	495 μ	478014457	687 μ
12602857	915 μ	45509137	697 μ	492387713	759 μ
12732529	327 μ	45721937	663 μ	499445449	733 μ
13001489	145 17	50088697	695 μ	504988801	897 μ
13068697	209 73	51909329	493 μ	508142377	751 μ
13974721	621 μ	52048313	519 μ	522026489	365 17
14042233	907 μ	52333297	377 μ	552784657	533 73
14160017	7453 μ	53152753	709 μ	563402449	785 μ
14414377	457 μ	53203889	955 μ	632133361	3125 μ
14751089	983 μ	58175849	319 89	635151689	849 μ
14896841	631 μ	60539593	213 17	701849009	775 μ
15290753	151 17	60665273	867 μ	793707041	985 μ
15499417	599 μ	65886001	943 μ	889334833	417 17
15601081	901 μ	66062657	793 μ	1094286241	9999 μ
16230041	191 41	68530937	469 μ		
17137793	385 μ	69593033	903 μ		
17522137	483 μ	87748937	429 μ		
17808841	921 μ	93621401	851 μ		
19007873	439 μ	95392169	541 μ		
20253553	391 μ	100104161	301 41		
23019641	511 μ	108003089	873 μ		
23754217	255 89	116490961	885 μ		
24840737	535 μ	119577209	953 μ		
24953633	633 μ	128307953	257 17		
27093617	803 μ	131579017	581 μ		
27766481	975 μ	135447881	375 73		
28271569	425 μ	144553441	2121 μ		
D 29423041	625 μ	183377633	281 17		

TAB. VII.

Octavan & Half-Octavan Primes.

	$p=x^8+y^8$	x, y		$p=\frac{1}{2}(x^8+y^8)$	x, y
$p<9.10^6$	257 65537 2070241	1, 2 1, 4 5, 6	$p<9.10^6$	1 198593	1, 1 3, 5
$p>9.10^6$	100006561 [None with $y \ngtr 38$]	3, 10 1, y	$p>9.10^6$	BJ 21523361 107182721 407865361	1, 9 3, 11 1, 13

High Prime Factors (p) *of Octavans.*

[μ=17, 41, 73, 89, 97, &c.]

	$p=\frac{1}{\mu}(x^8+y^8)$	y, μ		$p=\frac{1}{\mu}\cdot\frac{1}{2}(x^8+y^8)$	y, μ
L	22253377	64, μ	D	22191649	35, μ
	37642417	34, μ		29423041	25, μ
	57734881	26, μ		45534289	39, μ
	113607841	18, 97		70978049	79, μ
	164819521	122, μ		262965473	77, μ
	221201713	112, μ		291295393	71, μ
	291444977	54, μ		454677073	61, μ
	396622273	108, μ		502761569	155, μ
L	4278255361	32, μ			

TAB. VIII.

32^{io} *Residuacity of* 2 *with Quartan and Half-Quartan Primes.*

	$(2/p)_{32} = -1$			$(2/p)_{32} = +1$			
p	p	E	$x, \; y$	p	E	$x, \; y$	e
$p = x^4 + y^4$	257	256	1, 4	10657	32	9, 8	32
	2657	32	7, 4	65537	65536	1, 16	2^{11}
	54721	64	15, 8	83777	64	17, 4	64
	149057	64	17, 16	160001	256	1, 20	32
	166561	32	9, 20	243521	64	17, 20	64
	280097	32	23, 4	283937	32	23, 8	32
	334177	32	7, 24	331777	4096	1, 24	64
	614657	256	1, 28	394721	32	25, 8	32
	944257	128	31, 12	411361	32	25, 12	32
	1050977	32	7, 32	621217	32	9, 28	32
	1328417	32	23, 32	1055137	32	9, 32	32
	1682017	32	7, 36	1345921	128	33, 20	32
	1972097	128	31, 32	1763137	64	17, 36	64
	2070241	32	25, 36	1800577	128	33, 28	32
	3157537	32	41, 24	1959457	32	23, 36	32
	3874337	32	41, 32	2378977	32	39, 16	32
	4505377	32	41, 36	2473441	32	39, 20	32
	5039681	64	47, 20	2566561	32	9, 40	32
	5308417	65536	1, 48	2839841	32	23, 40	32
	5385761	32	41, 40	3362017	32	39, 32	32
	5785537	64	49, 12	4879937	64	47, 4	64
	7439681	64	47, 40	5211457	64	47, 24	64
	8324801	64	49, 40	5391937	64	17, 48	64
	8627777	64	47, 44	5768897	64	49, 8	32
	9834497	4096	1, 56	7591457	32	23, 52	32
	303595777	256	1, 132	40960001	65536	1, 80	512
	384160001	256	1, 140	59969537	4096	1, 88	32
				655360001	2^{20}	1, 160	32
$p = \frac{1}{2}(x^4 + y^4)$	67073	512	17, 15	1054721	2048	33, 31	64
	1416161	32	41, 9	2907713	64	49, 15	64
	2481601	64	47, 17	5473313	32	57, 25	32
	4715233	32	55, 23	5988193	32	55, 41	32
	8925313	128	65, 1	8388241	64	63, 31	32
				138461441	256	129, 1	32

THE CONVERSE OF FERMAT'S THEOREM.

By *E. B. Escott.*

IN the *Messenger of Mathematics*, Vol. XXVII. (1897-8), p. 174, is an article by Mr. J. H. Jeans, bearing the above title. The following method leads, however, more directly to the result, viz. to show that the converse of Fermat's theorem is not true, that is, if

$$e^{n-1}-1\equiv 0 \pmod{n},$$

it can be shown that this relation, which is always true when n is prime for any value of e prime to n, is for any particular value of e true for values of n which are not prime.

Let p_1, p_2, ... be prime factors of e^a-1 satisfying the condition $n=p_1p_2p_3...\equiv 1 \pmod{a}$. This can always be satisfied if e^a-1 contains two or more prime factors which do not divide a number of the same form with a smaller exponent than a. For, by Fermat's theorem, if p denote any such factor, a must divide $p-1$. Then $p\equiv 1 \pmod{a}$ and $p_1p_2\equiv 1 \pmod{a}$.

If the above condition is satisfied, it is obvious that the congruence at the beginning of this paper is satisfied.

Examples:

$$2^{10}-1=3.11.31.$$

Since 11 and 31 are both $\equiv 1$ (mod. 10), let $n=11.31=341$. Then

$$2^{340}-1\equiv 0 \pmod{341}.$$

In the following, let p be any prime factor of n, and let a be the smallest exponent of e, for which

$$e^a-1\equiv 0 \pmod{p}.$$

In $n=p_1p_2p_3...$ every p must be prime to every a, and n must be congruent to 1 (mod. a) for every a. If we have two prime factors p_1 and p_2 satisfying the first condition, that each prime factor p is prime to both the a's, then we can find a third factor p_3 among the factor of $e^{p_1p_2}-1$ which satisfies the condition $p_3\equiv 1$ (mod. a_1 and a_2).

Example: let $\quad e=2,\ p_1=3,\ p_2=5,$

then $\quad a_1=2,\ a_2=4.$

The factors of

$$2^{14}-1 \text{ are } 3,\ 43,\ 127.$$

$15p_3$ must be congruent to 1 (mod. 4). Therefore $p_3 \equiv 3$ (mod. 4), and both 43 and 127 satisfy the conditions. We have

$$2^{n-1} - 1 \equiv 0 \pmod{n},$$

for $n = 3.5.43$ or $3.5.127$.

In this way we find the following solutions of

$$2^{n-1} - 1 \equiv 0 \pmod{n}.$$

$n = 11.31$,	$n = 19.73$,	$n = 37.73$,	$n = 127.337$,	$n = 251.601$,
23.89,	29.113,	37.109,	89.397,	53.157,
43.127,	31.151,	73.109,	97.241,	257.641,
31.151,	31.331,	43.127,	97.673,	71.281.
17.257,	151.331,	43.337,	241.673,	

I give also a few examples where n has more than two factors:—

n=3.11.17,	n=3.5.29.43.113,
3.11.257,	3.5.29.113.127,
3.17.251,	3.11.29.31.43.281,
3.5.7.17,	3.11.31.71.127.281,
3.5.29.43,	7.11.13.31.41.61.331,
3.5.29.127,	7.11.13.31.41.61.151,
3.11.17.41,	7.11.13.31.41.61.151.1321,
3.11.31.127,	3.11.29.31.41.43.71.113.127.281,
3.11.41.157,	5.7.13.17.19.37.73.241.433.38737,
3.11.29.31.43,	3.11.29.31.41.43.71.113.127.281.122921,
3.5.43.113,	5.7.17.19.37.73.97.109.241.433.673.38737.

Consider the case where n has two or more equal prime factors

$$a^4 - 1 \equiv 0 \pmod{5^2},\quad a \equiv \pm 1,\ \pm 7 \pmod{5^2}.$$

In the same way as in the preceding case we may find solutions of $7^{n-1} - 1 \equiv 0 \pmod{n}$, where n contains the factor 5^2,

$n = 5^2.13$,
$5^2.73$,
$5^2.181$,
$5^2.193$,
$5^2.409$,
$5^2.13.181.811.1063$.

AN ELLIPSOIDAL TYPE OF ELLIPTIC INTEGRALS.

By *R. Hargreaves, M.A.*

If $d\omega$ is an element of area on unit sphere, perpendicular to a direction (l, m, n), and σ stands for $a^2l^2 + b^2m^2 + c^2n^2$, then the integrals $\int \sigma^{p-\frac{1}{2}} d\omega$, $\int (l^2, m^2, n^2)\, \sigma^{p-\frac{3}{2}} d\omega$, taken over the whole sphere, are algebraic if p is a negative integer; but if p is zero or a positive integer they are elliptic, *i.e.* they can be expressed in a linear manner in terms of the first and second elliptic integrals. We consider specially the cases $p = 0$ and 1, that is, the groups

$$4\pi S = \int \sigma^{-\frac{1}{2}} d\omega, \quad 4\pi (L, M, N) = \int (l^2, m^2, n^2)\, \sigma^{-\frac{3}{2}} d\omega \ldots \text{I.}\,(a, b),$$

$$4\pi S' = \int \sigma^{\frac{1}{2}} d\omega, \quad 4\pi (L', M', N') = \int (l^2, m^2, n^2)\, \sigma^{-\frac{1}{2}} d\omega \ldots \text{II.}\,(a, b).$$

They may be interpreted in connexion with the attraction of ellipsoids, and S is the inverse of the electric capacity of an ellipsoid.

We can pass from the original to a new ellipsoid by coupling Landen's transformation with the condition of unaltered capacity; the new axes are connected algebraically with the original, and if the transformation is repeated the movement is towards a prolate spheroid as final limit.

§ 1. The volume of an ellipsoid, when the element of volume is a cone with the centre as vertex, is given by $\frac{1}{3}\int r^3 d\omega$ or $\frac{1}{3}\int d\omega \Big/ \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)^{\frac{3}{2}}$, which is therefore equal to $4\pi abc/3$. Hence, altering the constants to give the above form of σ,

$$\frac{1}{4\pi}\int \frac{d\omega}{\sigma^{\frac{3}{2}}} = \frac{1}{abc} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From the definition of $SLMN$ and (1), we derive

$$La^2 + Mb^2 + Nc^2 = S, \quad L + M + N = 1/abc \;\ldots (2);$$

and these determine two, when the other two of the four integrals are known.

The ratios $a : b : c$ are connected with modulus and argument of the elliptic integrals by

$$k^2(c^2 - a^2) = c^2 - b^2, \quad a = c\cos\theta \ldots\ldots\ldots\ldots (3),$$

the order of magnitude being $c > b > a$. Thus

$$\sigma = c^2 \{l^2 \cos^2\theta + m^2 (1 - k^2 \sin^2\theta) + n^2\}$$
$$= c^2 \{1 - \sin^2\theta\, (l^2 + k^2 m^2)\} = c^2\sigma_1, \text{ say.}$$

Now $S\sqrt{(c^2 - a^2)}$ depends only on the ratios $a : b : c$, and we may write

$$F(k, \theta) = S\sqrt{(c^2 - a^2)} = \frac{\sqrt{(c^2 - a^2)}}{4\pi} \int \frac{d\omega}{\sigma^{\frac{1}{2}}} = \frac{\sin\theta}{4\pi} \int \frac{d\omega}{\sigma_1^{\frac{1}{2}}},$$

therefore

$$\frac{dF}{d\theta} = \frac{\cos\theta}{4\pi} \int \frac{d\omega}{\{l^2 \cos^2\theta + m^2 (1 - k^2 \sin^2\theta) + n^2\}^{\frac{3}{2}}} = \frac{1}{\sqrt{(1 - k^2 \sin^2\theta)}}$$

by (1).

Since $S\sqrt{(c^2 - a^2)}$ vanishes with θ, $F(k, \theta)$ is the first elliptic integral.

Again $N(c^2 - b^2)\sqrt{(c^2 - a^2)}$ depends only on the ratios $a : b : c$, and we may write

$$f(k, \theta) = N(c^2 - b^2)\sqrt{(c^2 - a^2)} = \frac{k^2 \sin^3\theta}{4\pi} \int \frac{n^2 d\omega}{\sigma_1^{\frac{3}{2}}},$$

and therefore

$$\frac{df}{d\theta} = \frac{3k^2 \sin^2\theta \cos\theta}{4\pi} \int \frac{n^2 d\omega}{\sigma_1^{\frac{5}{2}}}$$
$$= \frac{k^2 \sin^2\theta \cos\theta}{4\pi} \int \frac{1}{\cos^2\phi \cos^2\theta + \sin^2\phi\, (1 - k^2 \sin^2\theta)} \frac{d}{dn} \left(\frac{n^3}{\sigma_1^{\frac{3}{2}}}\right) d\omega.$$

In the last step we have taken the direction $n = 1$ as axis of a polar system, so that $l^2 = (1 - n^2)\cos^2\phi$, and $m^2 = (1 - n^2)\sin^2\phi$. Then σ_1 becomes

$$1 - \sin^2\theta (\cos^2\phi + k^2 \sin^2\phi)(1 - n^2)$$
$$= \cos^2\phi \cos^2\theta + \sin^2\phi\, (1 - k^2 \sin^2\theta) + n^2 \sin^2\theta\, (\cos^2\phi + k^2 \sin^2\phi).$$

At the same time $\int d\omega$ is $\int_0^{2\pi}\int_{-1}^{+1} d\phi\, dn$, and on integrating

$$\frac{df}{d\theta} = \frac{k^2 \sin^2\theta \cos\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{\cos^2\phi \cos^2\theta + \sin^2\phi\, (1 - k^2 \sin^2\theta)}$$
$$= \frac{k^2 \sin^2\theta}{\sqrt{(1 - k^2 \sin^2\theta)}} = \frac{1}{\sqrt{(1 - k^2 \sin^2\theta)}} - \sqrt{(1 - k^2 \sin^2\theta)}.$$

Since f vanishes with θ, $f = F(k, \theta) - E(k, \theta)$ the difference between first and second elliptic integrals. When (2) are solved to give L and M the whole scheme is:—

$$\left.\begin{aligned}
S &= \frac{1}{4\pi}\int \frac{d\omega}{\sigma^{\frac{1}{2}}} = \frac{F(k, \theta)}{(\sqrt{c^2 - a^2})} \\
L &= \frac{1}{4\pi}\int \frac{l^2 d\omega}{\sigma^{\frac{3}{2}}} = \frac{-E(k, \theta)}{(b^2 - a^2)\sqrt{(c^2 - a^2)}} + \frac{b}{ca\,(b^2 - a^2)} \\
M &= \frac{1}{4\pi}\int \frac{m^2 d\omega}{\sigma^{\frac{3}{2}}} = \frac{1}{\sqrt{(c^2 - a^2)}}\left\{E(k, \theta)\left(\frac{1}{c^2 - b^2} + \frac{1}{b^2 - a^2}\right)\right. \\
&\qquad\qquad \left. - \frac{F(k, \theta)}{c^2 - b^2}\right\} - \frac{a}{bc\,(b^2 - a^2)} \\
N &= \frac{1}{4\pi}\int \frac{n^2 d\omega}{\sigma^{\frac{3}{2}}} = \frac{F(k, \theta) - E(k, \theta)}{(c^2 - b^2)\sqrt{(c^2 - a^2)}}
\end{aligned}\right\}\ldots(4).$$

§ 2. To deal with the second group we trace their connexion with the first. Using $n = 1$ as polar axis,

$$l = \sqrt{(1 - n^2)}\cos\phi,\ m = \sqrt{(1 - n^2)}\sin\phi, \text{ and } d\omega = dn\,d\phi;$$

thus $(1 - n^2)\dfrac{\partial}{\partial n}(l^2, m^2, n^2,) = -2\,l^2 n,\ -2m^2 n,\ 2n(1 - n^2)$

respectively, and

$$(1 - n^2)\frac{\partial\sigma}{\partial n} = 2n\left\{-a^2 l^2 - b^2 m^2 + c^2(1 - n^2)\right\} = 2n\,(c^2 - \sigma)\ldots(5).$$

Thus

$$n^2\sigma^{-\frac{1}{2}} - c^2 n^2\sigma^{-\frac{3}{2}} = n\,(1 - n^2)\frac{\partial\sigma^{-\frac{1}{2}}}{\partial n} = \frac{\partial}{\partial n}n(1 - n^2)\,\sigma^{-\frac{1}{2}} + (3n^2 - 1)\,\sigma^{-\frac{1}{2}},$$

or
$$(1 - 2n^2)\,\sigma^{-\frac{1}{2}} - c^2 n^2\sigma^{-\frac{3}{2}} = \frac{\partial}{\partial n}\,n\,(1 - n^2)\,\sigma^{-\frac{1}{2}}.$$

Integrating this,

$$\left.\begin{aligned}
&\int_{-1}^{+1}\frac{(1 - 2n^2)\,dn}{\sigma^{\frac{1}{2}}} = \int_{-1}^{+1}\frac{c^2 n^2 dn}{\sigma^{\frac{3}{2}}}, \\
&\text{and therefore also} \\
&\int\frac{(1 - 2n^2)\,d\omega}{\sigma^{\frac{1}{2}}} = \int\frac{c^2 n^2 d\omega}{\sigma^{\frac{3}{2}}} \\
&\text{or} \qquad S - 2N' = c^2 N,
\end{aligned}\right\}\ldots\ldots\ldots\ldots(6),$$

with similar formulæ for $L'M'$, so that $L'M'N'$ are determined.

Now

$$\Sigma (a^2+b^2)(1-2n^2) = 2\Sigma a^2 - 2\Sigma a^2 \Sigma l^2 + 2\Sigma a^2 l^2 = 2\Sigma a^2 l^2 = 2\sigma;$$

thus from (6) we derive

$$\int \frac{\Sigma c^2 n^2 (a^2+b^2)}{\sigma^{\frac{3}{2}}} d\omega = 2\int \sigma^{\frac{1}{2}} d\omega,$$

or $$La^2(b^2+c^2) + Mb^2(c^2+a^2) + Nc^2(a^2+b^2) = 2S' \ldots (7).$$

Using the values of LMN in (4),

$$2S' = \frac{1}{2\pi}\int \sigma^{\frac{1}{2}} d\omega = \frac{ab}{c} + \frac{1}{\sqrt{(c^2-a^2)}}\{a^2 F(k,\theta) + (c^2-a^2)E(k,\theta)\} \ldots (8).$$

Also equations (2) and (7) may be used to express LMN in terms of S and S'.

Thus $$\left.\begin{aligned} L(a^2-b^2)(a^2-c^2) &= bc/a - 2S' + a^2 S \\ \text{and then} \qquad & \\ 2L'(a^2-b^2)(a^2-c^2) &= -abc + 2a^2 S' - (a^2b^2 + a^2c^2 - b^2c^2) S \end{aligned}\right\} \ldots (9).$$

follows by (6). We have now found the values of all the integrals in I. and II. in terms of $F(k, \theta)$ and $E(k, \theta)$, and have also expressed the associated integrals L, L' ... in terms of S and S', the principal integrals of ellipsoidal type.

§ 3. When $d\phi dn$ is written for $d\omega$ in $\int \sigma^{-\frac{1}{2}} d\omega$, the integration with regard to n can be carried out. Since

$$\sigma = (1-n^2)(a^2\cos^2\phi + b^2\sin^2\phi) + c^2 n^2$$
$$= n^2(c^2 - a^2\cos^2\phi - b^2\sin\phi^2) + a^2\cos^2\phi + b^2\sin^2\phi,$$

the result is

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \frac{1}{\sqrt{(c^2-a^2\cos^2\phi-b^2\sin^2\phi)}} \log\frac{c+\sqrt{(c^2-a^2\cos^2\phi-b^2\sin^2\phi)}}{c-\sqrt{(c^2-a^2\cos^2\phi-b^2\sin^2\phi)}} d\phi = \frac{F(k,\theta)}{\sqrt{(c^2-a^2)}}$$
(10a).

Treating S' in the same way, and using (10a),

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \sqrt{(c^2-a^2\cos^2\phi-b^2\sin^2\phi)} \log\frac{c+\sqrt{(c^2-a^2\cos^2\phi-b^2\sin^2\phi)}}{c-\sqrt{(c^2-a^2\cos^2\phi-b^2\sin^2\phi)}} d\phi$$
$$= \sqrt{(c^2-a^2)}\{F(k,\theta) - E(k,\theta)\} + c - ab/c \ldots (10b).$$

Here introduce k and θ, and use the notation

$$\Delta' = \sqrt{(1-k'^2\sin^2\phi)},$$

then

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \log\frac{1+\Delta'\sin\theta}{1-\Delta'\sin\theta}\frac{d\phi}{\Delta'} = F(k,\theta)\ldots\ldots(11a),$$

and

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \log\frac{1+\Delta'\sin\theta}{1-\Delta'\sin\theta}\Delta' d\phi$$

$$= F(k,\theta) - E(k,\theta) + \operatorname{cosec}\theta - \cot\theta\sqrt{(1-k^2\sin^2\theta)}\ldots(11b).$$

Another form of $(11a)$ is

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi} \log\frac{1+x\Delta'}{1-x\Delta'}\frac{d\phi}{\Delta'} = \int_0^x \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}\ldots(11a').$$

Differentiate this with regard to x, and on the left-hand we have

$$\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{1-x^2(1-k'^2\sin^2\phi)} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(1-x^2)\cos^2\phi+(1-k^2x^2)\sin^2\phi}$$

$$= \frac{1}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}},$$

so that the verification is immediate. [For $(11a)$ differentiate with regard to θ.]

When $(11a)$ is expanded,

$$\left.\begin{aligned} F(k,\theta) &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\left[\sin\theta+\frac{\sin^3\theta}{3}(1-k'^2\sin^2\phi)+\frac{\sin^5\theta}{5}(1-k'^2\sin^2\phi)^2+\ldots\right]d\phi \\ &= \sin\theta+\frac{\sin^3\theta}{3}(1-\tfrac{1}{2}k'^2)+\frac{\sin^5\theta}{5}(1-k'^2+\tfrac{3}{8}k'^4)+\ldots+\frac{u_n\sin^{2n+1}\theta}{2n+1}+\ldots, \\ \text{where}\quad u_n &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}(1-k'^2\sin^2\phi)^n d\phi, \\ \text{and } u_n \text{ is the coefficient in } &\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}} = 1+\ldots+u_nx^{2n}+\ldots \end{aligned}\right\}$$

$$\ldots(12).$$

For $k=1$, $u_n=1$; for $k=0$, $u_n=\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}$; as

$$F(1,\theta)=\tfrac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta},\text{ and } F(0,\theta)=\theta,$$

the coefficients are clearly correct in these cases. In general the coefficient u_n is a polynomial in k' or k, satisfying the

sequence equation

$$\left.\begin{aligned}&2(n+1)u_{n+1}-(2n+1)(1+k^2)u_n+2nk^2u_{n-1}=0,\\ &\text{and the differential equation}\\ &(1-k'^2)\frac{d^2u_n}{dk'^2}+\frac{1+(2n-2)k'^2}{k'}\frac{du_n}{dk'}+2nu_n=0\end{aligned}\right\}\ldots(13).$$

Very similar to (11) are the following:—

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\log\frac{\Delta'+k\sin\theta}{\Delta'-k\sin\theta}\frac{d\phi}{\Delta'}=F(k,\theta)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14a),$$

or $$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\log\frac{\Delta'+kx}{\Delta'-kx}\frac{d\phi}{\Delta'}=\int_0^x\frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}\ldots(14a'),$$

and

$$\frac{1}{\pi}\int_0^{\frac{1}{2}\pi}\log\frac{\Delta'+k\sin\theta}{\Delta'-k\sin\theta}\Delta'd\phi=F(k,\theta)-E(k,\theta)+k\sin\theta\ldots(14b).$$

They are at once verified by differentiation, as for (11). A comparison of the expansions for $F(k,\theta)$ given by (11a) and (14a) yields a formula, which in homogeneous form is

$$\int_0^{\frac{1}{2}\pi}\frac{d\phi}{(a^2\cos^2\phi+b^2\sin^2\phi)^{n+1}}=\frac{1}{a^{2n+1}b^{2n+1}}\int_0^{\frac{1}{2}\pi}(a^2\cos^2\phi+b^2\sin^2\phi)^n\,d\phi.$$

§ 4. We now deal with the change to a new ellipsoid with semi-axes $(a_1b_1c_1)$, for which S will remain unaltered, and the ratios of axes in the two cases will conform to Landen's transformation, viz.

$$k_1'=\frac{1-k}{1+k},\ \text{ and }\ k\sin\theta=\sin(2\theta_1-\theta)\ldots(15a,b).$$

The formulæ connecting the notations $(k\theta)$ and (abc) above are

$$\left.\begin{aligned}&k^2=\frac{c^2-b^2}{c^2-a^2},\quad k'^2=\frac{b^2-a^2}{c^2-a^2},\quad a=c\cos\theta;\\ &\text{and to these we add }\gamma=c\sin\theta,\ \gamma=\sqrt{(c^2-a^2)}\end{aligned}\right\}\ldots(16).$$

Landen's transformation makes $(1+k)F(k,\theta)=2F(k_1,\theta_1)$, and the constancy of S requires $F(k,\theta)/\gamma=F(k_1,\theta_1)/\gamma_1$. These are consistent, and the scale of magnitudes as well as

their ratios determined in the new scheme, if

$$\left.\begin{array}{l} 2\gamma_1 = \gamma(1+k), \\ \textit{i.e.} \quad 2\gamma_1 \text{ or } 2\sqrt{(c_1^2 - a_1^2)} = \sqrt{(c^2-a^2)} + \sqrt{(c^2-b^2)} \end{array}\right\} \text{...(17)}.$$

With this (15*a*) is $2\gamma_1 k_1' = \gamma(1-k)$, or

$$2\sqrt{(b_1^2 - a_1^2)} = \sqrt{(c^2-a^2)} - \sqrt{(c^2-b^2)} \text{ ....(18)}.$$

Also (15*b*) is

$$(1+k)\sin\theta = 2\sin\theta_1\cos(\theta_1 - \theta),$$

or

$$\gamma_1 \sin\theta = \gamma\sin\theta_1\cos(\theta_1 - \theta),$$

or

$$c_1 = c\cos(\theta_1 - \theta) = (aa_1 + \gamma\gamma_1)/c_1,$$

or

$$aa_1 + \gamma\gamma_1 = c_1^2 = \gamma_1^2 + a_1^2,$$

i.e.

$$4(a_1^2 - a_1 a) = 4\gamma_1(\gamma - \gamma_1)$$
$$= \{\sqrt{(c^2-a^2)} + \sqrt{(c^2-b^2)}\}\{\sqrt{(c^2-a^2)} - \sqrt{(c^2-b^2)}\} = b^2 - a^2,$$

and

$$2a_1 = a + b \text{ ....(19)}.$$

With the aid of (19), (17) gives

$$\left.\begin{array}{ll} & 2c_1^2 = c^2 + ab + \sqrt{\{(c^2-a^2)(c^2-b^2)\}}, \\ \text{or} & 2c_1 = \sqrt{\{(c+a)(c+b)\}} + \sqrt{\{(c-a)(c-b)\}}; \\ \text{and (18) gives} & \\ & 2b_1^2 = c^2 + ab - \sqrt{\{(c^2-a^2)(c^2-b^2)\}}, \\ \text{or} & 2b_1 = \sqrt{\{(c+a)(c+b)\}} - \sqrt{\{(c-a)(c-b)\}} \end{array}\right\} \text{...(20)}.$$

We note also the relations

$$b_1^2 + c_1^2 = c^2 + ab,\ 2b_1c_1 = c(a+b) = 2ca_1,\ 4a_1b_1c_1 = c(a+b)^2 \text{...(21)}.$$

The solution for a movement in the contrary direction from $a_1b_1c_1$ to abc is

$$a = a_1 - \frac{\sqrt{\{(c_1^2 - a_1^2)(b_1^2 - a_1^2)\}}}{a_1},\quad b = b_1 + \frac{\sqrt{\{(c_1^2-a_1^2)(b_1^2-a_1^2)\}}}{a_1},\quad c = \frac{c_1b_1}{a_1}$$
$$\text{......(22)}.$$

We now connect the values of S', the second ellipsoidal integral, before and after transformation. For this we quote (8), viz.

$$2S' = \frac{ab}{c} + a^2S + \gamma E(k, \theta),$$

and Landen's formula

$$(1+k)\,E(k_1,\,\theta_1)=E(k,\,\theta)+k\sin\theta-\tfrac{1}{2}(1-k^2)\,F(k,\,\theta),$$

which, since $2\gamma_1=\gamma(1+k)$, is

$$2\gamma_1 E_1=\gamma E+\gamma k\sin\theta-\tfrac{1}{2}\gamma^2(1-k^2)\,S.$$

Thus

$$4S_1'-2S'=\frac{2a_1b_1}{c_1}-\frac{ab}{c}+(2a_1^2-a^2)\,S+2\gamma_1E_1-\gamma E$$

$$=\frac{1}{c}\left[2b_1^2-ab+\sqrt{\{(c^2-a^2)(c^2-b^2)\}}\right]+\{2a_1^2-a^2-\tfrac{1}{2}(b^2-a^2)\}\,S$$

$$=c+abS \quad\ldots\ldots\ldots\ldots\ldots\ldots(23).$$

Thus the transformation which leaves S unaltered also makes the formula connecting the second integrals very simple.

§ 5. We now give details as to the trend of the movement, when the transformation is repeated. The original cba being positive magnitudes in descending order, otherwise unrestricted, $c_1b_1a_1$ have a like character with the restriction $a_1^2>\dfrac{b_1^2c_1^2}{b_1^2+c_1^2}$, which is the condition that 'a' should be positive when the transformation is reversed. Also $a<a_1$ and $c>c_1$, or the extremes are brought nearer together. If $a=b$ the transformation becomes an identity, a fact which points to a special position as regards the prolate spheroid. If $c=b$ (oblate spheroid) the transformation is $2c_1=a+c$, $2c_1^2=c\,(c+a)$, or $c_1^2=ca_1$; *i.e.* successive use is made of arithmetic and geometric means (not quite the same as in Gauss's theorem). In terms of k and θ this case has $k=0$, and therefore $2\theta_1=\theta$ or θ is halved at each step, therefore ultimately $\cos\theta=1$ and $a=c$, *i.e.* the limit is a sphere. For the oblate spheroid

$$S=\theta/\sqrt{(c^2-a^2)},\quad 2S'=a+c^2S,$$

where $a=c\cos\theta$. For the prolate spheroid

$$S=\frac{1}{2\sqrt{(c^2-a^2)}}\log\frac{c+\sqrt{(c^2-a^2)}}{c-\sqrt{(c^2-a^2)}},\quad 2S'=c+a^2S.$$

In (15*a*) the movement of k does not depend on θ, and takes place in the sense of increasing k and ultimately making k' indefinitely small $\left[k_1'=\dfrac{1-k}{1+k}=k'\times\sqrt{\left\{\dfrac{1-k}{(1+k)^3}\right\}}\right]$;

the movement is rapid unless k is very small originally. Now k' small implies b and a nearly equal, or the movement is to the prolate spheroid as limit. As

$$c_1^2 - b_1^2 = \sqrt{\{(c^2-a^2)(c^2-b^2)\}} > c^2 - b^2$$

is an inequality true for each transformation, the three axes are not all brought together unless $c = b$, *i.e.* for the oblate spheroid as original form. In (15b) θ is less than 90°, because $a = c\cos\theta$, and a and c are both positive, therefore θ is diminished at each step; but as k approaches 1, the movement of θ becomes very small. If dashes are used for limiting values, then θ', which determines the ratio $a' : c'$ is given by

$$\left.\begin{aligned} \tfrac{1}{2}\log\frac{1+\sin\theta'}{1-\sin\theta'} &= F(0,\theta') \\ &= F(k,\theta)\times\tfrac{1}{2}(1+k)\times\tfrac{1}{2}(1+k_1)\times\tfrac{1}{2}(1+k_2)\times\ldots, \\ \text{or} \quad &= F(k,\theta)\times\sqrt{k}\div\sqrt{(k_1k_2k_3)}\ldots \end{aligned}\right\} \quad (23),$$

and the magnitude of c' or a' is then given by

$$\frac{1}{2\sqrt{(c'^2-a'^2)}}\log\frac{1+\sin\theta'}{1-\sin\theta'} = S, \text{ or } = \frac{F(k,\theta)}{\sqrt{(c^2-a^2)}}\ldots(24),$$

the left-hand member being the ultimate form of S.

Successive approximations to the value of the product of factors depending on k in (23) are $\frac{1}{2}(1+k)$, $\frac{1}{4}\{(1+k)(1+k_1)\}$, ... in excess, and $\sqrt{k}$, $\sqrt{(k/k_1)}$, $\sqrt{(k/k_1k_2)}$, ... in defect; their differences being

$$\tfrac{1}{2}(1-\sqrt{k})^2,\ \tfrac{1}{4}(1+k)(1-\sqrt{k_1})^2,\ \tfrac{1}{8}(1+k)(1+k_1)(1-\sqrt{k_2})^2,$$

in succession. Dividing (23) by (24),

$$\sqrt{(c'^2-a'^2)} = \sqrt{(c^2-b^2)}\div\sqrt{(k_1k_2k_3)}\ldots.$$

§ 6. The more general forms of integral indicated at the outset may be briefly discussed, *i.e.* the integrals

$$4\pi S_p = \int\sigma^{p-\frac{1}{2}}d\omega, \text{ and } 4\pi(L_p, M_p, N_p) = \int(l^2, m^2, n^2)\,\sigma^{p-\frac{3}{2}}d\omega \quad \ldots(25).$$

If in (5) we form $\dfrac{\partial\sigma^{p-\frac{1}{2}}}{\partial n}$ instead of $\dfrac{\partial\sigma^{-\frac{1}{2}}}{\partial n}$, we get

$$2(p+1)L_{p+1} = (2p-1)a^2L_p + S_p\ldots\ldots\ldots(26).$$

Also $\Sigma L_p = S_{p-1}$, and $\Sigma a^2 L_p = S_p$ ......(27*a*, *b*),

precisely as in (2). The addition of the several forms in (26) gives a result which is obvious when (27) is used; but addition with multipliers (a^2, b^2, c^2), or (a^{-2}, b^{-2}, c^{-2}) gives new forms, viz.

$$(2p-1)\,\Sigma a^4 L_p = 2\,(p+1)\,S_{p+1} - S_p \Sigma a^2 \text{ ......(28)},$$

and

$$2p\Sigma a^{-2} L_p = (2p-3)\,S_{p-2} + S_{p-1}\Sigma a^{-2} \text{ ....(29)},$$

where in the latter p has been depressed by 1. The solution of (27*a*, *b*) and (28) gives

$$L_p(a^2-b^2)(a^2-c^2) = b^2c^2 S_{p-1} - (b^2+c^2)\,S_p + \{2(p+1)\,S_{p+1} - S_p\Sigma a^2\}/(2p-1) \text{ ...(30)},$$

which makes

$$(2p-1)\,\Sigma b^2c^2 L_p = (2p-1)\,S_{p-1}\Sigma a^2b^2 - 2pS_p\Sigma a^2 + 2\,(p+1)\,S_{p+1}.$$

On comparing this with (29) we have the sequence equation for S_p, viz.

$$4p\,(p+1)\,S_{p+1} - 4p^2 S_p\,(a^2+b^2+c^2)$$
$$+ (2p-1)^2 S_{p-1}(a^2b^2+b^2c^2+c^2a^2) - (2p-1)(2p-3)\,S_{p-2}a^2b^2c^2 = 0 \text{...(31)}.$$

For positive values of p, S_p is given in terms of S_1, S_0, and S_{-1}, or of S', S, and $1/abc$. For negative values of p all are referred back to S_{-1} or $1/abc$, *i.e.* S_{-2} is given in terms of S_{-1}, then S_{-3} in terms of S_{-2} and S_{-1}, after which the full sequence equation appears. Notwithstanding the change from transcendental to algebraic type, there is no exception to the validity of the equations.

§ 7. In conclusion a few words may be added on the interpretation of the integrals.

If in (1) we write $a^2+\lambda$, ... for $a^2b^2c^2$, but retain the use of σ for $a^2l^2+b^2m^2+c^2n^2$, we have

$$\frac{1}{4\pi}\int\frac{d\omega}{(\sigma+\lambda)^{\frac{3}{2}}} = \frac{1}{\sqrt{\{(a^2+\lambda)\,(b^2+\lambda)\,(c^2+\lambda)\}}}.$$

Integrating with regard to λ,

$$S \text{ or } \frac{1}{4\pi}\int\frac{d\omega}{\sigma^{\frac{1}{2}}} = \tfrac{1}{2}\int_0^\infty \frac{d\lambda}{\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}} \text{...(32}a\text{)}.$$

and differentiating with regard to a^2,

$$L \text{ or } \frac{1}{4\pi}\int\frac{l^2 d\omega}{\sigma^{\frac{3}{2}}} = \tfrac{1}{2}\int_0^\infty \frac{d\lambda}{(a^2+\lambda)\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}} \text{...(32}b\text{)}.$$

We are thus brought into touch with the attraction integrals, and the evaluation in terms of elliptic integrals given above corresponds to the expressions used by Darwin in his paper, "on Jacobi's figure of equilibrium for a rotating mass of fluid," *Proc. Roy. Soc.*, Nov. 25, 1886.

Thus S, as already stated, is the capacity of an ellipsoidal conductor of semi-axes abc. With respect to the constancy of S in the transformation, it will be observed that the new ellipsoid has the greater volume, for (21) gives

$$4a_1b_1c_1 = c(a+b)^2 = 4abc + c(a-b)^2,$$

but is more spherical. With given charge these changes tend to produce opposite changes in the energy or in the capacity, which are here balanced.

With uniform volume-density ρ, the internal potential of an ellipsoid is

$$V_i = \pi\rho abc\int_0^\infty \frac{d\lambda}{\sqrt{\{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}}}\left(1-\frac{x^2}{a^2+\lambda}-\frac{y^2}{b^2+\lambda}-\frac{z^2}{c^2+\lambda}\right)$$

$$= 2\pi\rho abc\,(S - Lx^2 - My^2 - Nz^2)$$

$$= \frac{\rho abc}{2}\int\left(1-\frac{l^2x^2+m^2y^2+n^2z^2}{\sigma}\right)\frac{d\omega}{\sigma^{\frac{1}{2}}},$$

or $$2\pi V_i = \pi\rho abc\int\left\{1-\frac{(lx+my+nz)^2}{\sigma}\right\}\frac{d\omega}{\sigma^{\frac{1}{2}}} \quad\text{......(33).}$$

For the external potential $\sigma+\lambda$ appears in place of σ, where

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1.$$

As regards geometrical interpretation, if p is the perpendicular drawn in the direction (lmn) from the centre to a tangent plane, $p^2=\sigma$, and the area of a central section of the ellipsoid parallel to the tangent plane is $\pi abc/\sigma^{\frac{1}{2}}$; the area of a parallel section through the point xyz is

$$\frac{\pi abc}{\sigma^{\frac{1}{2}}}\left\{1-\frac{(lx+my+nz)^2}{\sigma}\right\}.$$

If A denotes this area, $2\pi V_i = \rho\int A\,d\omega$ is the interpretation of (33), and this is a particular case of a general theorem given in *Phil. Mag.*, April 1905, p. 465.

The transformation of S to $(l'm'n')$ where $l' = al/\sigma^{\frac{1}{2}}, \ldots$, so that $\sigma^{-1} = \Sigma l'^2/a^2$ is of some interest. The Jacobian

$$\frac{\partial(n', \phi')}{\partial(n, \phi)} = \frac{abc}{\sigma^{\frac{3}{2}}}, \text{ or } d\omega' = \frac{abc}{\sigma^{\frac{3}{2}}} d\omega;$$

thus

$$\int \left(\frac{l'^2}{a^2} + \frac{m'^2}{b^2} + \frac{n'^2}{c^2}\right)^{-1} d\omega' = \int \sigma d\omega' = abc \int \frac{d\omega}{\sigma^{\frac{1}{2}}},$$

and also

$$\int \frac{l'^2}{a^2} \left(\frac{l'^2}{a^2} + \frac{m'^2}{b^2} + \frac{n'^2}{c^2}\right)^{-1} d\omega' = abc \int \frac{l^2 d\omega}{\sigma^{\frac{3}{2}}}.$$

In Thomson and Tait's *Nat. Phil.* Pt. II., § 494*l*, the integral is evaluated in an unsymmetrical way.

GROUP OF ORDER p^6 WHICH DOES NOT INCLUDE AN ABELIAN SUBGROUP OF ORDER p^4.

By *G. A. Miller*.

FROM the general theorem that every group of order p^m contains an abelian subgroup of order p^α, whenever m is greater than $\frac{1}{2}\{\alpha(\alpha-1)\}$, it follows that every group of order p^7 contains an abelian subgroup of order p^4.* It has been observed that this theorem admits of being extended, since every group of order 2^6 also contains an abelian subgroup of order 2^4.† The present note is devoted to a proof of the theorem that for every odd value of p there is a group of order p^6 which does not contain an abelian subgroup of order p^4. Hence $p = 2$ is the only case where the given theorem admits of extension for groups of order p^6.

Consider the abelian group of order p^3 and of type $(1, 1, 1)$, which is generated by t_1, t_2, t_3; and let t_1 be invariant under the group G of order p^6, which involves this group of order p^3. We select another operator (t_4) of order p such that $t_4^{-1}t_2t_4 = t_2$, $t_4^{-1}t_3t_4 = t_1t_3$. The group generated by these four operators $\{t_1, t_2, t_3, t_4\} \equiv \{t_2, t_3, t_4\}$ is of order p^4 and involves no operator whose order exceeds p, when $p > 2$, as we shall assume in what follows. This group of order p^4 is extended by means of t_5, which satisfies the following conditions: $t_5^p = 1$, $t_5^{-1}t_2t_5 = t_2$,

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* *Messenger of Mathematics*, Vol. XXVII. (1898), p. 119.

† *Ibid.*, p. 79 of the present volume.

$t_5^{-1}t_3t_5 = t_2t_3$, $t_5^{-1}t_4t_5 = t_3t_4$. We thus obtain a group of order p^5 which has p^2 invariant operators, while each of its other operators has p^2 conjugates. Hence every abelian subgroup of order p^3 involves the p^2 invariant operators, and there are p^2+p+1 such subgroups. The quotient group with respect to the invariant operators is the non-abelian group of order p^3, which contains no operator of order p^2.

To obtain the group of order p^6 in question we extend the group of order p^5 which has just been constructed by means of t_6, which has the following properties: $t_6^{-1}t_2t_6 = t_1t_2$, $t_6^{-1}t_3t_6 = t_3$, $t_6^{-1}t_4t_6 = t_4$, $t_6^{-1}t_5t_6 = t_5$. If this resulting group of order p^6 would involve an abelian subgroup of order p^4, the subgroup would have p^3 operators in common with the group of order p^5 considered above. This is impossible, since every abelian subgroup of order p^3 in this group of order p^5 involves t_2, while none of the remaining operators are commutative with t_2. Hence $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ is a group of order p^6 which does not contain any Abelian subgroup of order p^4.

ON THE COMPLETE SOLUTION IN INTEGERS, FOR CERTAIN VALUES OF p, OF $a(a^2+pb^2) = c(c^2+pd^2)$.

By *H. Holden.*

1. THE method of solution employed depends on the assumption that any factor of a, or c, not of the form l^2+pm^2, is common to both. This will be true, if for the determinant $-p$ there is only one properly primitive class, and that if there are improperly primitive classes the highest power of 2, which divides l^2+pm^2, is an even one. These conditions are satisfied for $p = 1, \pm 2, 3, -5, -13, -29, -53$, and -61.

Let e be this common factor, then we may put

$$a = e(l_\alpha^2 + pm_\alpha^2)(l_\theta^2 + pm_\theta^2),$$

$$c = e(l_\gamma^2 + pm_\gamma^2)(l_\theta^2 + pm_\theta^2),$$

where $l_\alpha^2 + pm_\alpha^2$ and $l_\gamma^2 + pm_\gamma^2$ are relatively prime.

In consequence, we have

$$a^2 + pb^2 = (l_\gamma^2 + pm_\gamma^2)(l_\varphi^2 + pm_\varphi^2),$$

$$c^2 + pd^2 = (l_\alpha^2 + pm_\alpha^2)(l_\varphi^2 + pm_\varphi^2),$$

from which, remembering that l_α, m_α, l_γ, m_γ, l_ϕ, m_ϕ may independently be changed to $-l_\alpha$, $-m_\alpha$, etc., we have

$$a = l_\gamma l_\phi + pm_\gamma m_\phi,$$

$$b = l_\gamma m_\phi - m_\gamma l_\phi,$$

$$c = l_\alpha l_\phi + pm_\alpha m_\phi,$$

$$d = l_\alpha m_\phi - m_\alpha l_\phi.$$

For the sake of brevity, let α be written for $l_\alpha^2 + pm_\alpha^2$; γ, θ, and ϕ denoting similar expressions.

Equating the two values of a and c,

$$l_\gamma l_\phi + pm_\gamma m_\phi = e\alpha\theta,$$

$$l_\alpha l_\phi + pm_\alpha m_\phi = e\gamma\theta.$$

Solving for l_ϕ and m_ϕ,

$$l_\phi = \frac{e(\alpha\theta m_\alpha - \gamma\theta m_\gamma)}{l_\gamma m_\alpha - l_\alpha m_\gamma},$$

$$m_\phi = \frac{e(\gamma\theta l_\gamma - \alpha\theta l_\alpha)}{p(l_\gamma m_\alpha - l_\alpha m_\gamma)}.$$

Substituting these values in the expressions for b and d,

$$b = \frac{e(\gamma\theta l_\gamma^2 - \alpha\theta l_\alpha l_\gamma - p\alpha\theta m_\alpha m_\gamma + p\gamma\theta m_\gamma^2)}{p(l_\gamma m_\alpha - l_\alpha m_\gamma)}$$

$$= \frac{e\{\gamma^2\theta - \alpha\theta(l_\alpha l_\gamma + pm_\alpha m_\gamma)\}}{p(l_\gamma m_\alpha - l_\alpha m_\gamma)},$$

and $$d = -\frac{e\{\alpha^2\theta - \gamma\theta(l_\alpha l_\gamma + pm_\alpha m_\gamma)\}}{p(l_\gamma m_\alpha - l_\alpha m_\gamma)}.$$

If $L_{\alpha\gamma}$ and $M_{\alpha\gamma}$ satisfy the equation $L_{\alpha\gamma}^2 + pM_{\alpha\gamma}^2 = \alpha\gamma$, we have, remembering that the signs of b and d are immaterial,

$$a = e\alpha\theta,$$

$$b = \frac{e(\gamma^2\theta - \alpha\theta L_{\alpha\gamma})}{pM_{\alpha\gamma}},$$

$$c = e\gamma\theta,$$

$$d = \frac{e(\alpha^2\theta - \gamma\theta L_{\alpha\gamma})}{pM_{\alpha\gamma}}.$$

Multiplying throughout by $pM_{a\gamma}$, and dividing by $e\theta$, we get all the integral values of a, b, c, and d satisfying the given equation. Since $-L_{a\gamma}$ may equally well replace $L_{a\gamma}$, these values are

$$a = p\alpha.M_{a\gamma},$$

$$b = \gamma^2 + \alpha.L_{a\gamma},$$

$$c = p\gamma.M_{a\gamma},$$

$$d = \alpha^2 + \gamma.L_{a\gamma}.$$

From any assumed set of values for l_a, m_a, l_γ, m_γ we generally get at least four different values of a, b, c, and d, for there will be two corresponding values of $L_{a\gamma}$ and $M_{a\gamma}$, and b and d will have different values according as $L_{a\gamma}$ is taken positively or negatively. Or, for any values of a and c satisfying the given equation, there will, as a rule, be two numerically different values of b and d which may be used.

2. The following results may be deduced:—

(*a*) The complete solutions of

$$a(pa^2+b^2) = c(pc^2+d^2)$$

are given by

$$a = \alpha M_{a\gamma},$$

$$b = \gamma^2 + \alpha L_{a\gamma},$$

$$c = \gamma M_{a\gamma},$$

$$d = \alpha^2 + \gamma.L_{a\gamma}.$$

(*b*) For $p=3$, the equation $a(a^2+pb^2)=c(c^2+pd^2)$ is equivalent to $(a+b)^3+(a-b)^3=(c+d)^3+(c-d)^3$, and hence as no assumption has been made as to the relative magnitudes of a and b, or c and d, the above solutions completely solve $x^3+y^3+z^3+t^3=0$, where one or two cubes may be negative.

(*c*) No real solution is possible for which $b=0$.

For, if $b=0$, we have $\gamma^2+\alpha L_{a\gamma}=0$, and, as α and γ must be relatively prime in order that a, b, c, and d may be so, it follows that $\alpha=1$, or $\gamma=1$, and either of these alternatives necessitates the other, and leads to no real solution.

Thus, no integral solutions exist of

$$a^3 = c(c^2+pd^2),$$

or for $p=3$ of

$$2a^3=(c+d)^3+(c-d)^3,$$

that is of $$x^3+y^3=2z^3.$$

(d) No real solution exists for which $a=b$.

This is easily proved in the same way, and so there are no integral solutions of

$$(p+1)\,a^3=c\,(c^2+pd^2),$$

or for $p=3$ of

$$(2a)^3=(c+d)^3+(c-d)^3,$$

or of $$x^3+y^3=z^3.$$

3. In a similar manner it may be proved that the complete integral solutions of

$$c\,(a^2+pb^2)=a\,(c^2+pd^2),$$

for the stated values of p, are given by

$$a=p\alpha M_{\alpha\gamma},$$
$$b=\gamma\alpha+\alpha L_{\alpha\gamma},$$
$$c=p\gamma M_{\alpha\gamma},$$
$$d=\gamma\alpha+\gamma L_{\alpha\gamma}.$$

Similarly, if $R=r\,(l_\rho^2+pm_\rho^2)$ and $S=s\,(l_\sigma^2+pm_\sigma^2)$ are given quantities, the complete solution in integers, for the stated values of p, of $Ra\,(a^2+pb^2)=Sc\,(c^2+pd^2)$ is given by

$$a=pk\alpha M_{a\rho.\gamma\sigma},$$
$$b=l\sigma\gamma^2+k\alpha L_{a\rho.\gamma\sigma},$$
$$c=pl\gamma M_{a\rho.\gamma\sigma},$$
$$d=k\rho\alpha^2+l\gamma L_{a\rho.\gamma\sigma},$$

where k and l are any values satisfying the equation $kr=ls$.

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END OF VOL. XXXVI.

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